67. On a Generalization of MacPherson's Chern Homology Class. III

By Shoji YOKURA*)
Department of Mathematics, College of Liberal Arts,
University of Kagoshima

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§0. Introduction. In [3] Deligne-Grothendieck-MacPherson's natural transformation C_* is the unique natural transformation from the "constructible function" covariant functor $\mathcal{F}$ to the usual Z-homology covariant functor $H_*(\ ;Z)$, satisfying the extra condition that if X is smooth then the special value $C_*(X)(\mathbf{1}_X)$ is equal to the Poincaré dual of the total Chern cohomology class $c(X)$ of the variety X, where $\mathbf{1}_X$ is the characteristic function on X. If the above extra condition is dropped, then there are infinitely many natural transformations from $\mathcal{F}$ to $H_*(\ ;Z)$. Indeed, if we let $C_{*i}:\mathcal{F}\to H_{2i}(\ ;Z)$ be the composite of the above natural transformation $C_*:\mathcal{F}\to H_*(\ ;Z)$ and $H_*(\ ;Z)\to H_{2i}(\ ;Z)$, the natural transformation "picking up" the $2i$-dimensional component of the total homology class, then the linear form $\sum_{i\geq 0} m_i C_{*i}$ is obviously a natural transformation, where m_i is an integer. Thus there is a simple problem of determining all natural transformations from $\mathcal{F}$ to $H_*(\ ;Z)$, to which there is a naive conjecture (due to G. Kennedy): *any natural transformation must be the above linear form* $\sum_{i\geq 0} m_i C_{*i}$. This conjecture is still open. In this paper we announce a characterization of this linear natural transformation $\sum_{i\geq 0} m_i C_{*i}$, which can be proved using the same technique as that of our previous papers [5, 6] and we give a very partial result to this conjecture. Detailed proofs will appear somewhere else.

§1. A characterization of $\sum_{i\geq 0} m_i C_{*i}$. The linear natural transformation $\sum_{i\geq 0} m_i C_{*i}$ obviously satisfies the condition that $(\sum_{i\geq 0} m_i C_{*i})(V)(\mathbf{1}_V) = (\sum_{0\leq i\leq \dim V} m_{\dim V - i} c_i)(V)\cap[V]$ *for any compact complex smooth variety* V. In fact, we can show that any natural transformation $\tau:\mathcal{F}\to H_*(\ ;Z)$ satisfying this kind of extra condition must be linear. To be more precise, let us call $cl^{(n)}=\sum_{0\leq i\leq n} cl_i$ a *degree-n characteristic class* of complex vector bundles, where $cl_0=\lambda_0 c_0$ and $cl_i=P_i(c_1, c_2, \cdots, c_i)$ is a homogeneous polynomial of degree i with k-th Chern class c_k being of weight k. Note that any characteristic cohomology class of a complex vector bundle can be expressed as a polynomial of individual Chern classes ([1, 4]). With this terminology we can show the following

Theorem 1.1. *Let* $\{cl^{(n)}\}_{n\geq 0}$ *be a sequence of degree-n characteristic*

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classes. A necessary and sufficient condition for that $\tau: \mathcal{F} \to H_*(\ ; Z)$ *is a natural transformation satisfying the "dimension-wise universal" condition that* $\tau(V)(1_V) = cl^{(\dim V)}(V) \cap [V]$ *for any compact complex smooth variety V is that there exists a sequence of integers* $\{m_i\}_{i \geqq 0}$ *such that each* $cl^{(n)} = \sum_{0 \leqq i \leqq n} m_{n-i} c_i$, i.e., $\tau = \sum_{i \geqq 0} m_i C_{*i}$.

For a *total characteristic class* $cl = \sum_{i \geqq 0} cl_i$ let $cl^{[n]} = \sum_{0 \leqq i \leqq n} cl_i$ be the *n-th truncated characteristic class* of the total characteristic class *cl* (cf. [2, Appendice B]), which is a degree-n characteristic class. Since $cl(V) = cl^{[\dim V]}(V)$, we get the following

Corollary 1.2 ([5, Theorem (1.4)]). *Let* $cl = \sum_{i \geqq 0} cl_i$ *be a total characteristic class of complex vector bundles. A necessary and sufficient condition for that* $\tau: \mathcal{F} \to H_*(\ ; Z)$ *is a natural transformation satisfying the "universal" condition that* $\tau(V)(1_V) = cl(V) \cap [V]$ *for any compact complex smooth variety V is that cl is an integral multiple of the total Chern class c*, i.e., $cl = m(\sum_{i \geqq 0} c_i)$, i.e., $\tau = mC_*$.

The above theorem can be proved by using "linear independence of Chern numbers" (for a more precise statement of this, see Milnor's book [4]) and the following propositions.

Proposition 1.3. *Let* $I_k(n) = \{r_1, r_2, \cdots, r_k\}$ *be a partition of n and let* $c_{I_k(n)} := c_{r_1} \cdot c_{r_2} \cdots c_{r_k}$. *For compact complex manifolds X and Y, where* $\dim Y = n$, *and the projection* $\pi: X \times Y \to X$, *the following equality holds:* $\pi_*(c_{I_k(n)}(X \times Y) \cap [X \times Y]) = (c_{I_k(n)}(Y) \cap [Y])[X]$.

Proposition 1.4. *Suppose that* τ *and* $\tau': \mathcal{F} \to H_*(\ ; Z)$ *are two natural transformations. Then* $\tau = \tau'$ *if and only if* $\tau(V)(1_V) = \tau'(V)(1_V)$ *for any compact complex smooth variety V.*

As a modified version of Kennedy's conjecture, we pose the following one:

Conjecture. *If* $\tau: \mathcal{F} \to H_*(\ ; Z)$ *is a natural transformation such that for any compact complex smooth variety V* $\tau(V)(1_V)$ *is the Poincaré dual of a characteristic cohomology class of the variety V, then* τ *must be linear*, i.e., $\tau = \sum_{i \geqq 0} m_i C_{*i}$.

§ 2. A partial result about the conjecture. At the moment we can show only the following partial results about the conjecture.

Proposition 2.1. *Let* $\tau: \mathcal{F} \to H_*(\ ; Z)$ *be a natural transformation.*

(1) *There exists a unique integer* m_0 *such that for any compact complex (not necessarily smooth) variety X*

$$(\tau(X)(1_X))_0 = m_0 C_{*0}(X)(1_X) = m_0 \chi(X).$$

(2) *There exists a unique integer* m_n *such that for any compact complex smooth variety X of each dimension n*

$$(\tau(X)(1_X))_{2n} = m_n C_{*n}(X)(1_X) = m_n [X].$$

Here $(\cdot)_i$ *means the i-dimensional component of the total homology class.*

Corollary 2.2. *Let* $\tau: \mathcal{F} \to H_*(\ ; Z)$ *be a natural transformation. If we restrict this natural transformation* τ *to the subcategory of compact complex algebraic varieties of dimension* $\leqq 1$, *then there exist integers* m_0

and m_1 *such that* $\tau = m_0 C_{*0} + m_1 C_{*1}$.

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