64. An Additive Problem of Prime Numbers. II

By Akio FUJII
Department of Mathematics, Rikkyo University

(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1991)

1. Introduction. We continue our study [1] on the mean value of

$$r_2(n)=\sum_{m+k=n}\Lambda(m)\Lambda(k),$$

where $\Lambda(x)=\log p$ if $x=p^m$ with a prime number p and an integer $m\geq 1$, and $=0$ otherwise. We put

$$S_2(n)=\prod_{p|n}\left(1+\frac{1}{p-1}\right)\prod_{p\nmid n}\left(1-\frac{1}{(p-1)^2}\right).$$

Then in [1] we have shown under the Riemann Hypothesis (R.H.) that

$$\sum_{n\leq X}(r_2(n)-nS_2(n))=O(X^{3/2}).$$

Here we shall clarify the oscillating nature of the right hand side of this formula. We shall carry it out by expressing several places of the previous arguments in [1] more explicitly. As a result we shall prove the following theorem.

Theorem (Under R.H.). *For $X>X_0$, we have*

$$\sum_{n\leq X}(r_2(n)-nS_2(n))=-4X^{3/2}\Re\left[\sum_{\gamma>0}\frac{X^{i\gamma}}{((1/2)+i\gamma)((3/2)+i\gamma)}\right]+O((X\log X)^{1+(1/3)}),$$

where γ runs over the imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$.

As is seen below, another oscillating nature connected with the distribution of the zeros of $\zeta(s)$ is hidden in the remainder term

$$O((X\log X)^{1+(1/3)}),$$

although the bounded quantity

$$G(X)\equiv\Re\left[\sum_{\gamma>0}\frac{X^{i\gamma}}{((1/2)+i\gamma)((3/2)+i\gamma)}\right]$$

represents the main oscillation.

We assume R.H. throughout this article.

Some information concerning the quantity $G(X)$ will be noticed in the forthcoming article [2].

2. Proof of Theorem. We put

$$R(y)=\sum_{n\leq y}\Lambda(n)-y \qquad \text{for } y\geq 0.$$

We put $N=[X]$. Then we have

$$\sum_{n\leq X}r_2(n)=\sum_{m\leq X}\Lambda(m)(X-m)+\sum_{2\leq m\leq X-2}\Lambda(m)(R(X-m)-R(N-m))$$
$$+\sum_{2\leq m\leq N-2}\Lambda(m)R(N-m)+O(\log X)$$
$$=S_1+S_2+S_3+O(\log X), \text{ say.}$$

We get first

$$S_1=\int_2^X (\sum_{m\le y} \Lambda(m))\,dy=\frac{1}{2}X^2+\int_2^X R(y)\,dy+O(1)$$
$$=\frac{1}{2}X^2-2X^{3/2}\Re\Big(\sum_{\gamma>0}\frac{X^{i\gamma}}{((1/2)+i\gamma)((3/2)+i\gamma)}\Big)+O(X),$$
where we may notice, without assuming any unproved hypothesis, that for $X>1$,
$$\int_0^X (\sum_{n\le y}\Lambda(n)-y)\,dy=-\sum_{\substack{\zeta(\rho)=0\\ 0<\Re(\rho)<1}}\frac{X^{\rho+1}}{\rho(\rho+1)}-X\sum_{a=1}^{\infty}\frac{X^{-2a}}{2a(2a-1)}$$
$$-\frac{\zeta'}{\zeta}(0)X+\frac{\zeta'}{\zeta}(-1).$$
We see next that
$$S_2=\sum_{2\le m\le X-2}\Lambda(m)\Big(\sum_{N-m\le n\le X-m}\Lambda(n)-(X-N)\Big)\ll X\log X.$$
To treat S_3, we decompose it first as follows.
$$S_3=\sum_{2\le m\le N-2}\Lambda(N-m)R(m)=\sum_{X^{3/4}\le m\le N-2}\Lambda(N-m)R(m)+O(X^{9/8}\log^2 X).$$
Now suppose that $\frac{1}{2}\Big(\frac{X}{\log^2 X}\Big)^{2/3}\equiv\frac{1}{2}Y\le T\le Y\le\frac{1}{2}X^{3/4}$. To get an explicit formula for the last sum, we shall apply pp. 406–412 of Wolke [5]. We put
$$g(m,T)=\sum_{n\le m}\Lambda(n)-\frac{1}{2}\Lambda(m)-\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}\Big(-\frac{\zeta'}{\zeta}(s)\Big)\frac{m^s}{s}\,ds$$
with $\varepsilon=\frac{1}{\log X}$.

Then $g(m,T)$ is a continuous function of T and satisfies
$$g(m,T)=R(m)-\frac{1}{2}\Lambda(m)+\sum_{|\gamma|\le T}\frac{m^{(1/2)+i\gamma}}{(1/2)+i\gamma}+O\Big(\frac{m}{T}\Big).$$
Moreover we have
$$\frac{1}{Y}\int_{(1/2)Y}^{Y}\sum_{X^{3/4}\le m\le N-2}\Lambda(N-m)g(m,T)\,dT$$
$$\ll\frac{1}{Y^2}\sum_{X^{3/4}\le m\le N-2}\Lambda(N-m)\sum_{k\le m-(m/Y)}\frac{\Lambda(k)}{\log^2(m/k)}$$
$$+\sum_{X^{3/4}\le m\le N-2}\Lambda(N-m)\sum_{m-(m/Y)<k\le m+(m/Y)}\Lambda(k)$$
$$+\frac{1}{Y^2}\sum_{X^{3/4}\le m\le N-2}\Lambda(N-m)\sum_{m+(m/Y)<k\le 2m}\frac{\Lambda(k)}{\log^2(k/m)}$$
$$+\frac{1}{Y^2}\sum_{X^{3/4}\le m\le N-2}m\Lambda(N-m)\sum_{k>2m}\frac{\Lambda(k)}{k^{1+\varepsilon}\log^2(k/m)}\ll\frac{X^2}{Y}.$$
Thus we can find T in the interval $(Y/2)\le T\le Y$ such that
$$S_3=-\sum_{X^{3/4}\le m\le N-2}\Lambda(N-m)\sqrt{m}\sum_{|\gamma|\le T}\frac{m^{i\gamma}}{(1/2)+i\gamma}+O\Big(\frac{X^2}{T}\Big)$$
$$=-2\Im\Big\{\sum_{2\le m\le N-2}\sqrt{m}\Lambda(N-m)\sum_{0<\gamma\le T}\frac{m^{i\gamma}}{\gamma}\Big\}$$
$$+\frac{1}{2}\sum_{2\le m\le N-2}\sqrt{m}\Lambda(N-m)\sum_{|\gamma|\le T}\frac{m^{i\gamma}}{((1/2)+i\gamma)i\gamma}+O\Big(\frac{X^2}{T}\Big)$$

$$=S_4+S_5+O\left(\frac{X^2}{T}\right), \text{ say.}$$

We fix this T below.

We shall evaluate S_5 first.

$$S_5=\frac{1}{2}\sum_{|\gamma|\le T}\frac{1}{((1/2)+i\gamma)i\gamma}\sum_{2\le m\le N-2}\sqrt{m}\Lambda(N-m)m^{i\gamma}$$
$$=\frac{1}{2}\sum_{|\gamma|\le T}\frac{1}{((1/2)+i\gamma)i\gamma}\int_1^N y^{i\gamma}d\left(\frac{2}{3}y^{3/2}-\sqrt{y}R(N-y)+\frac{1}{2}\int_{N-y}^{N-1}\frac{R(u)}{\sqrt{N-u}}du\right)$$
$$=\frac{1}{2}\sum_{|\gamma|\le T}\frac{1}{((1/2)+i\gamma)i\gamma}\int_1^X y^{(1/2)+i\gamma}dy+\frac{1}{2}\sum_{|\gamma|\le T}\frac{1}{((1/2)+i\gamma)}\int_1^N y^{i\gamma}\frac{R(N-y)}{\sqrt{y}}dy$$
$$+\frac{1}{4}\sum_{|\gamma|\le T}\frac{1}{((1/2)+i\gamma)i\gamma}\int_1^N y^{i\gamma}\frac{R(N-y)}{\sqrt{y}}dy+O(\sqrt{X}\log^2 X)$$
$$=\frac{1}{2}X^{3/2}\sum_{|\gamma|\le T}\frac{X^{i\gamma}}{((1/2)+i\gamma)((3/2)+i\gamma)i\gamma}$$
$$+O\left(\sum_{|\gamma|\le T}\frac{1}{|\gamma|}\sqrt{\int_1^N R^2(N-y)dy}\sqrt{\int_1^N\frac{dy}{y}}\right)+O(\sqrt{X}\log^2 X)$$
$$=\frac{1}{2}X^{3/2}\sum_{|\gamma|\le T}\frac{X^{i\gamma}}{((1/2)+i\gamma)((3/2)+i\gamma)i\gamma}+O(X\log^{5/2}X),$$

where we have used the estimate

$$\int_2^X\frac{R^2(y)}{y}dy\ll X.$$

Using the same argument as in [1], we get

$$S_4=-2\Im\left(-\int_1^T\int_0^{\log N}e^{x((3/2)+it)}\frac{1}{2\pi t}\log\frac{t}{2\pi}dt\,dx\right.$$
$$+\int_1^T\int_0^{\log N}e^{x((3/2)+it)}\,d\left(\sum_{0<\gamma\le t}\frac{1}{\gamma}\right)dx$$
$$\left.+\int_1^T\int_0^{\log N}e^{xit}\frac{1}{2\pi t}\log\frac{t}{2\pi}d\left(\sum_{m\le e^x}\sqrt{m}\Lambda(N-m)\right)dt\right)$$
$$+O\left(T\log X\left(\frac{X}{\sqrt{T}}\log X+\left(\frac{X}{T}\right)^{3/2}+\frac{X}{T}\log^2 X\right)\right)$$
$$=S_6+S_7+S_8+O\left(T\log X\left(\frac{X}{\sqrt{T}}\log X+\left(\frac{X}{T}\right)^{3/2}+\frac{X}{T}\log^2 X\right)\right), \text{ say.}$$

To get the last remainder term has been the main subject of our previous [1]. We should notice that Gallagher's lemma [3] has been used for this purpose.

$$S_6=\frac{1}{\pi}X^{3/2}\Im\left(\int_1^T\frac{X^{it}}{(3/2)+it}\frac{1}{t}\log\frac{t}{2\pi}dt\right)+O\left(\sqrt{X}\log^2 X\right).$$

$$S_7=-2X^{3/2}\Im\left(\sum_{0<\gamma\le T}\frac{X^{i\gamma}}{((3/2)+i\gamma)\gamma}\right)+O(\sqrt{X}\log^2 X).$$

$$S_8=-\frac{1}{\pi}\Im\left\{\int_1^{\log N}\left(\frac{e^{xiT}}{T}\log\frac{T}{2\pi}-e^{xi}\log\frac{1}{2\pi}\right)\frac{1}{ix}d\left(\sum_{m\le e^x}\sqrt{m}\Lambda(N-m)\right)\right.$$
$$\left.-\int_1^{\log N}\int_1^T\frac{e^{xit}}{ixt^2}\left(1-\log\frac{t}{2\pi}\right)d\left(\sum_{m\le e^x}\sqrt{m}\Lambda(N-m)\right)\right\}+O(\log^3 X)$$

$$= -\frac{1}{\pi}\Im\{S_9+S_{10}\}+O(\log^3 X), \text{ say.}$$

$$S_9=\int_1^{\log N}\left(\frac{e^{xiT}}{T}\log\frac{T}{2\pi}-e^{xi}\log\frac{1}{2\pi}\right)\frac{e^{3x/2}}{ix}dx$$
$$+\left[\left(\frac{e^{xiT}}{T}\log\frac{T}{2\pi}-e^{xi}\log\frac{1}{2\pi}\right)\frac{1}{ix}(-\sqrt{e^x}R(N-e^x))\right]_1^{\log N}$$
$$+\frac{1}{i}\int_1^{\log N}\sqrt{e^x}R(N-e^x)\left\{-\frac{1}{x^2}\left(\frac{e^{xiT}}{T}\log\frac{T}{2\pi}-e^{xi}\log\frac{1}{2\pi}\right)\right.$$
$$\left.+\frac{1}{x}\left(ie^{xiT}\log\frac{T}{2\pi}-ie^{xi}\log\frac{1}{2\pi}\right)\right\}dx$$
$$+\frac{1}{2i}\int_1^{\log N}\frac{1}{x}\sqrt{e^x}R(N-e^x)\left(\frac{e^{xiT}}{T}\log\frac{T}{2\pi}-e^{xi}\log\frac{1}{2\pi}\right)dx$$
$$=\frac{\log(2\pi)}{i}\int_1^{\log X}\frac{e^{((3/2)+i)x}}{x}dx+O\left(\frac{X^{3/2}}{T}\log T\log X\right)+O(X\log X).$$

Hence, we get

$$-\frac{1}{\pi}\Im\{S_9\}=\frac{\log(2\pi)}{\pi}\Re\{Li(X^{(3/2)+i})\}+O\left(\frac{X^{3/2}}{T}\log T\log X\right)+O(X\log X).$$

Similarly, we get

$$S_{10}=-\int_1^{\log N}\int_1^T\frac{e^{xit}}{ixt^2}\left(1-\log\frac{t}{2\pi}\right)e^{3x/2}\,dt\,dx$$
$$-\int_1^T\left[\frac{e^{xit}}{ixt^2}\left(1-\log\frac{t}{2\pi}\right)(-\sqrt{e^x}R(N-e^x))\right]_1^{\log N}dt$$
$$+\int_1^{\log N}\int_1^T(-\sqrt{e^x}R(N-e^x))\frac{e^{xit}}{ixt^2}\left(1-\log\frac{t}{2\pi}\right)\left(-\frac{1}{x}+it\right)dt\,dx$$
$$-\frac{1}{2}\int_1^{\log N}\int_1^T\sqrt{e^x}R(N-e^x)\frac{e^{xit}}{ixt^2}\left(1-\log\frac{t}{2\pi}\right)dt\,dx$$
$$=\frac{1}{i}\int_1^T\frac{1}{t^2}\log\frac{t}{2\pi e}\int_1^{\log N}\frac{e^{x((3/2)+it)}}{x}dx\,dt+O(X\log^2 X).$$

Hence, we get

$$-\frac{1}{\pi}\Im\{S_{10}\}=\frac{1}{\pi}\int_1^T\frac{1}{t^2}\log\frac{t}{2\pi e}\Re\{Li(X^{(3/2)+it})\}dt+O(X\log^2 X).$$

By our choice of T, we get

$$\sum_{n\le X}r_2(n)=\frac{1}{2}X^2-2X^{3/2}\Re\left\{\sum_{\gamma>0}\frac{X^{i\gamma}}{((1/2)+i\gamma)((3/2)+i\gamma)}\right\}$$
$$+\frac{1}{2}X^{3/2}\sum_{|\gamma|\le(X/\log^2 X)^{2/3}}\frac{X^{i\gamma}}{((1/2)+i\gamma)((3/2)+i\gamma)i\gamma}$$
$$+\frac{1}{\pi}X^{3/2}\Im\left\{\int_1^{(X/\log^2 X)^{2/3}}\frac{X^{it}}{(3/2)+it}\frac{1}{t}\log\frac{t}{2\pi}dt\right\}$$
$$-2X^{3/2}\Im\left\{\sum_{0<\gamma\le(X/\log^2 X)^{2/3}}\frac{X^{i\gamma}}{((3/2)+i\gamma)\gamma}\right\}+\frac{\log(2\pi)}{\pi}\Re\{Li(X^{(3/2)+i})\}$$
$$+\frac{1}{\pi}\int_1^{(X/\log^2 X)^{2/3}}\frac{1}{t^2}\log\frac{t}{2\pi e}\Re\{Li(X^{(3/2)+it})\}dt+O((X\log X)^{1+(1/3)})$$
$$=\frac{1}{2}X^2-4X^{3/2}\Re\left\{\sum_{\gamma>0}\frac{X^{i\gamma}}{((1/2)+i\gamma)((3/2)+i\gamma)}\right\}$$

$$+\frac{1}{\pi}X^{3/2}\Re\left\{\int_1^\infty \frac{X^{it}}{((3/2)+it)it}\log\frac{t}{2\pi}dt\right\}+\frac{\log(2\pi)}{\pi}\Re\{Li(X^{(3/2)+i})\}$$
$$+\frac{1}{\pi}\int_1^\infty \frac{1}{t^2}\log\frac{t}{2\pi e}\Re\{Li(X^{(3/2)+it})\}dt+O((X\log X)^{1+(1/3)})$$
$$=\frac{1}{2}X^2-4X^{3/2}\Re\left\{\sum_{\gamma>0}\frac{X^{i\gamma}}{((1/2)+i\gamma)((3/2)+i\gamma)}\right\}+O((X\log X)^{1+(1/3)}).$$

This proves our Theorem, since

$$\sum_{n\le X} nS_2(n)=\frac{1}{2}X^2+O(X\log X).$$

References

[1] A. Fujii: An additive problem of prime numbers (to appear in Acta Arith., LVIII (1991)).
[2] ——: An additive problem of prime numbers. III (to appear).
[3] P. X. Gallagher: A double sum over primes and zeros of the zeta function. Number Theory, Trace Formula and Discrete Groups. Academic Press, pp. 229–240 (1989).
[4] K. Prachar: Primzahlverteilung. Springer (1957).
[5] D. Wolke: On the explicit formula of Riemann-von Mangoldt. II. J. London Math. Soc., (2), **28**, 406–416 (1983).