54. A Note on Multivalent Functions

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(Communicated by Shokichi IYANAGA, M. J. A., June 11, 1991)

1. Introduction. Let $A_p(n)$ be the class of functions of the form

$$f(z)=z^p+\sum_{k=p+n}^{\infty} a_k z^k \quad (p \in N=\{1,2,3,\cdots\};\ n \in N) \tag{1.1}$$

which are analytic in the open unit disk $U=\{z: |z|<1\}$. A function $f(z) \in A_p(n)$ is said to be in the class $A_p(n,\alpha)$ if it satisfies

$$\left|\frac{f(z)}{z^p}-1\right|<1-\alpha \tag{1.2}$$

for some $\alpha (0 \leq \alpha < 1)$ and for all $z \in U$.

Recently, Saitoh [3] has studied the class $A_p(n,\alpha)$ and proved some properties for functions belonging to $A_p(n,\alpha)$. Our main result in this paper contains a result due to Saitoh [3, Theorem 1].

2. Main result. We derive the main result by using the following lemma due to Miller and Mocanu [2] (also, due to Jack [1]).

Lemma. *Let $w(z)=w_n z^n+w_{n+1}z^{n+1}+\cdots$ be regular in U with $w(z) \not\equiv 0$ and $n \geq 1$. If $z_0=r_0 e^{i\theta_0}$ $(r_0<1)$ and*

$$|w(z_0)|=\sum_{|z|\leq r_0} |w(z)| \tag{2.1}$$

then

$$z_0 w'(z_0)=m w(z_0) \tag{2.2}$$

and

$$\mathrm{Re}\left(1+\frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq m, \tag{2.3}$$

where $m \geq n \geq 1$.

Theorem. *If $f(z) \in A_p(n)$ with $f(z) \not\equiv z^p$ satisfies*

$$\left|\beta\frac{f(z)}{z^p}+\gamma\frac{f'(z)}{z^{p-1}}-(\beta+p\gamma)\right|<(1-\alpha)\{\beta+(p+n)\gamma\} \tag{2.4}$$

for some $\alpha (0 \leq \alpha < 1)$, $\beta (\beta \geq 0)$, $\gamma (\gamma \geq 0)$, $\beta+\gamma>0$, and for all $z \in U$, then $f(z) \in A_p(n,\alpha)$.

Proof. Defining the function $w(z)$ by

$$\frac{f(z)}{z^p}-1=(1-\alpha)\omega(z) \tag{2.5}$$

for $f(z) \in A_p(n)$, we see that $w(z)=w_n z^n+w_{n+1}z^{n+1}+\cdots$ is regular in U and $w(z) \not\equiv 0$. It follows from (2.5) that

$$\frac{f'(z)}{z^{p-1}}=p+(1-\alpha)\{pw(z)+zw'(z)\}. \tag{2.6}$$

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1990 Mathematics Subject Classifications. 30C45.

Therefore, we have

$$\beta\frac{f(z)}{z^p}+\gamma\frac{f'(z)}{z^{p-1}}-(\beta+p\gamma)=(1-\alpha)\{(\beta+p\gamma)w(z)+\gamma zw'(z)\}. \tag{2.7}$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1.$$

Then, applying the lemma and letting $w(z_0)=e^{i\theta_0}$, we obtain

$$\left|\beta\frac{f(z_0)}{z_0^{\,p}}+\gamma\frac{f'(z_0)}{z_0^{\,p-1}}-(\beta+p\gamma)\right|=(1-\alpha)(\beta+p\gamma+m\gamma)\quad(m\geq n\geq 1) \tag{2.8}$$

$$\geq(1-\alpha)\{\beta+(p+n)\gamma\},$$

which contradicts with our condition (2.4). This shows that $|w(z)|<1$ for all $z \in U$, that is,

$$\left|\frac{f(z)}{z^p}-1\right|<1-\alpha\quad(z\in U). \tag{2.9}$$

This completes the proof of the theorem.

Letting $\gamma=\beta$ in Theorem, we have

Corollary 1. *If $f(z)\in A_p(n)$ with $f(z)\not\equiv z^p$ satisfies*

$$\left|\frac{f(z)}{z^p}+\frac{f'(z)}{z^{p-1}}-(p+1)\right|<(1-\alpha)(p+n+1)\quad(z\in U) \tag{2.10}$$

for some $\alpha(0\leq\alpha<1)$, then $f(z)\in A_p(n,\alpha)$.

Making $\beta=1-(p+n)\gamma$, Theorem leads to

Corollary 2. *If $f(z)\in A_p(n)$ with $f(z)\not\equiv z^p$ satisfies*

$$\left|\{1-(p+n)\gamma\}\frac{f(z)}{z^p}+\gamma\frac{f'(z)}{z^{p-1}}-(1-n\gamma)\right|<1-\alpha \tag{2.11}$$

for some $\alpha(0\leq\alpha<1)$, $\gamma(0\leq\gamma<1/(p+n))$ and for all $z\in U$, then $f(z)\in A_p(n,\alpha)$.

Further, taking $\gamma=1-\beta$ in Theorem, we have

Corollary 3 ([3]). *If $f(z)\in A_p(n)$ with $f(z)\not\equiv z^p$ satisfies*

$$\left|\beta\left(\frac{f(z)}{z^p}-1\right)+(1-\beta)\left(\frac{f'(z)}{z^{p-1}}-p\right)\right|<(1-\alpha)\{(p+n)-(p+n-1)\beta\} \tag{2.12}$$

for some $\alpha(0\leq\alpha<1)$, $\beta(0\leq\beta\leq1)$, and for all $z\in U$, then $f(z)\in A_p(n,\alpha)$.

References

[1] I. S. Jack: Functions starlike and convex of order α. J. London Math. Soc., **3**, 469–474 (1971).
[2] S. S. Miller and P. T. Mocanu: Second order differential inequalities in the complex plane. J. Math. Anal. Appl., **65**, 289–305 (1978).
[3] H. Saitoh: On certain class of multivalent functions (preprint).