50. L^p Estimate for Abstract Linear Parabolic Equations

By Mariko GIGA,*) Yoshikazu GIGA,**) and Hermann SOHR***)

(Communicated by Kunihiko KODAIRA, M. J. A., June 11, 1991)

§ 1. **Introduction.** We are interested in existence and a priori estimate of solutions of parabolic equations

$$(1.1)\qquad \begin{cases} du/dt+A(t)u=f \\ u(0)=a \end{cases} \qquad 0\leq t<T\leq\infty \qquad f\in L^q(0, T; X).$$

in a Banach space X by using the method of pure imaginary power $A(t)^{is}$.

The case that A is independent of t is already investigated. In [1] Dore and Venni proved that when A has a bounded inverse the Cauchy problem (1.1) has a unique solution u for given $f\in L^q(0, T; X)$ and $a=0$ such that

$$(1.2)\qquad \int_0^T \|u'(t)\|^q dt+\int_0^T \|Au(t)\|^q dt\leq C\int_0^T \|f(t)\|^q dt$$

where $C=C(T, q)$, provided the following conditions are satisfied:

(1.3) X is a ζ-convex Banach space equipped with the norm $\|\cdot\|$,

(1.4) $\|A^{is}\|\leq Ke^{\theta|s|}$ for all $s\in \boldsymbol{R}$ where $0\leq\theta<\pi/2$.

For the notion of ζ-convexity see [1] and the references cited there.

In [2] Sohr and Y. Giga extended this theory to the case that A need not have a bounded inverse and they showed that (1.2) holds with C independent of T; see also [3] for another proof. Furthermore they applied this a priori estimate to the Navier-Stokes equations.

The aim of this note is to extend their result to the case that A depends on time t. We show the existence and a priori estimate of solutions of (1.1) in the case $A=A(t)$ depends on t; at least when the domain of $A(t)$, $\mathcal{D}(A(t))$ is independent of t.

Our result here is different and does not follow the solvability results in Tanabe [4], Yagi [5, 6] because (i) our solution satisfies an L^p estimate and (ii) we assume less regularity for f and $A(t)A(0)^{-1}$. On the other hand, (1.3) and (1.4) are stronger conditions than the analyticity assumption in [4, 5, 6] (see [3]).

§ 2. **Main result.** Let X be a complex ζ-convex Banach space and $0<T\leq\infty$. $\mathcal{L}(X)$ denotes the space of bounded linear operators in X.

We consider operators $A(t)$ defined in X for $0\leq t<T$ satisfying:

(2.1) a) For $0\leq t<T$, $A(t)$ is a closed linear operator, the domain $\mathcal{D}(A(t))$ and the range $\mathcal{R}(A(t))$ of $A(t)$, are dense in X and the null space $N(A(t))$ is zero.

*) School of General Education, Nippon Medical School.
**) Department of Mathematics, Hokkaido University.
***) Department of Mathematics, University of Paderborn, Germany.

b) For $0\leq t\leq T$ and $\tau>0$ we have $(\tau+A(t))^{-1}\in\mathcal{L}(X)$ and there are constants $M(t)>0$ such that $\|(\tau+A(t))^{-1}\|\leq M(t)/\tau$ for $0\leq t<T$, $\tau>0$; $\|\cdot\|$ denotes the operator norm.

c) The pure imaginary powers $A(t)^{is}$ are in $\mathcal{L}(X)$ for all $0\leq t<T$ and $s\in\mathbf{R}$. There are constants $K>0$, $0\leq\theta<\pi/2$ independent of t and s such that $\|A(t)^{is}\|\leq Ke^{\theta|s|}$ for $0\leq t<T$, $s\in\mathbf{R}$.

d) The domain of $A(t)$ does not depend on t; so we write $\mathcal{D}(A)$ instead of $\mathcal{D}(A(t))$. There is a positive constant C such that $\|A(t)x\|\leq C\|A(\tau)x\|$ for $0\leq\tau\leq t<T$ and $x\in\mathcal{D}(A)$; it follows that (the closure of) $A(t)A(\tau)^{-1}\in\mathcal{L}(X)$ for $0\leq\tau\leq t<T$ and $\|A(t)\cdot A(\tau)^{-1}\|\leq C$.

e) The map $t, \tau\to A(t)A(\tau)^{-1}$ is continuous from $\{(\tau,t): 0\leq\tau\leq t<T\}$ to $\mathcal{L}(X)$ where $\mathcal{L}(X)$ is equipped with the operator norm.

f) If $T=\infty$, then $\lim_{t>\tau\to\infty}A(t)A(\tau)^{-1}=I$ with respect to the operator norm where I is the identity.

Before discussing existence and a priori estimate of solutions to (1.1), we consider the appropriate space of initial values a. Let $1<q<\infty$, $0\leq t<T\leq\infty$. We define

$$\mathcal{T}_t^q=\left\{a\in X: \|a\|_{\mathcal{T}_t^q}=\left(\int_t^T\|A(t)e^{-(\tau-t)A(t)}a\|^q d\tau\right)^{1/q}<\infty\right\}. \tag{2.2}$$

Remark. (i) We know (see [3]) from the assumption (2.1) that each $-A(t)$ generates an analytic bounded semigroup $\{e^{-\tau A(t)}: \tau>0\}$ with $\|e^{-\tau A(t)}\|\leq C$, $\|A^\alpha(t)e^{-\tau A(t)}\|\leq C/\tau^\alpha$ $(\alpha\geq 0)$. Using this estimates we can show that $\mathcal{D}(A)\cap\mathcal{R}(A(t))\subseteq\mathcal{T}_t^q$ for $0\leq t<T$. We know also that $\mathcal{D}(A)\cap\mathcal{R}(A(t))$ is dense in X.

(ii) $\mathcal{T}_t^q$ is a normed space but not a Banach space in general; it becomes a Banach space when we add $\|a\|$ on the right in (2.2). However, we can extend the theory given here to more general initial values by using the completion of $\mathcal{T}_t^q$ under the norm above.

We state the main theorem. We denote $\dot{u}=du/dt$; $L^q(0,T;X)$ is the space of all measurable $f:[0,T]\to X$ with $\|f\|_{L^q(0,T;X)}=\left(\int_0^T\|f\|^q dt\right)^{1/q}<\infty$.

Theorem. *Let X be a complex ζ-convex Banach space and let $1<q<\infty$, $0<T\leq\infty$. Suppose $f\in L^q(0,T;X)$ and $a\in\mathcal{T}_0^q$. Then under the assumption* (2.1) a)–f), *there exists a unique measurable function $u:[0,T)\to X$ with the following properties.*

i) $\int_0^T\|\dot{u}\|^q d\tau<\infty$, *$u(\tau)\in\mathcal{D}(A)$ for* a.e. $\tau\in[0,T)$ *and* $\int_0^T\|A(\tau)u(\tau)\|^q d\tau<\infty$,

ii) $\dot{u}(\tau)+A(\tau)u(\tau)=f(\tau)$ *and* $u(0)=a$ *for* a.e. $\tau\in[0,T)$,

iii) $\int_0^T\|\dot{u}\|^q d\tau+\int_0^T\|A(\tau)u(\tau)\|^q d\tau\leq C\left[\|a\|_{\mathcal{T}_0^q}^q+\int_0^T\|f(\tau)\|^q d\tau\right]$

where C is independent of a and f. In particular, if $T=\infty$, we obtain

$$\int_0^\infty\|\dot{u}\|^q d\tau+\int_0^\infty\|A(\tau)u(\tau)\|^q d\tau\leq C\left[\|a\|_{\mathcal{T}_0^q}^q+\int_0^\infty\|f(\tau)\|^q d\tau\right].$$

§ 3. Proof of the theorem.

We introduce the function space:

$$W_t^q=\Big\{u:[t,T)\longrightarrow X:\ u \text{ measurable},\ u(\tau)\in\mathcal{D}(A)\ \text{for a.e. } \tau\in[t,T),$$
$$\int_t^T \|A(t)u(\tau)\|^q d\tau<\infty \quad \int_t^T \|\dot u\|^q d\tau<\infty\Big\},$$
$$\|u\|_{W_t^q}=\Big(\int_t^T \|A(t)u(\tau)\|^q d\tau\Big)^{1/q}+\Big(\int_t^T \|\dot u\|^q d\tau\Big)^{1/q} \qquad 0\le t<T.$$

We also introduce the trace space at t:
$$F_t^q=\{u(t):\ u\in W_t^q\} \qquad 0\le t<T,$$
with the quotient norm $\|a\|_{F_t^q}=\inf\{\|u\|_{W_t^q}:\ u\in W_t^q\ u(t)=a\}$.

An essential part of the proof is the following lemma. In the following, $C_1, C_2, C_3, \cdots$ are positive constants whose values are not specified.

Lemma 1. i) *It holds $\mathcal{T}_t^q=F_t^q$ and the norms $\|a\|_{\mathcal{T}_t^q}$, $\|a\|_{F_t^q}$ are equivalent.*

ii) *There exists a constant C such that $\|u\|_{\mathcal{T}_t^q}\le C\|u\|_{W_t^q}$.*

iii) *For each $a\in\mathcal{T}_t^q$ there exists some extension $u\in W_t^q$ with $a=u(t)$ and $\|u\|_{W_t^q}\le C\|a\|_{\mathcal{T}_t^q}$ where $C>0$ is independent of a. Such an extension is given by $u(\tau)=e^{-(\tau-t)A(t)}a$ for $t\le\tau<T$.*

Proof. First we observe that $\mathcal{T}_t^q\subseteq F_t^q$. To show this, let $a\in\mathcal{T}_t^q$ and put $u(\tau)=e^{-(\tau-t)A(t)}a$. Then it follows easily form the definition that $\|a\|_{F_t^q}\le \|u\|_{W_t^q}=2\|a\|_{\mathcal{T}_t^q}<\infty$. Thus we have $a\in F_t^q$.

Next we show the converse direction that $F_t^q\subseteq\mathcal{T}_t^q$. Let $a\in F_t^q$ and $u(\tau)=a$ with $u\in W_t^b$. Then we have the representation
$$u(\tau)=e^{-(\tau-t)A(t)}a+\int_t^\tau e^{-(\tau-s)A(t)}[\dot u(s)+A(t)u(s)]ds.$$
We put
$$u_1(\tau)=\int_t^\tau e^{-(\tau-s)A(t)}[\dot u+A(t)u]ds.$$
Then we see that $u_1(t)=0$ and $\dot u_1(s)+A(t)u_1(s)=\dot u(s)+A(t)u(s)$ for $t\le s<T$. From the L^q estimate when A is independent of τ (see [2]) we see that
$$\int_t^T \|\dot u_1\|^q ds<\infty, \qquad \int_t^T \|A(t)u_1(s)\|^q ds<\infty$$
which means that $u_1\in W_t^q$. Setting $u_2(\tau)=e^{-(\tau-t)A(t)}a$ we obtain $u_2(\tau)=u(\tau)-u_1(\tau)$ for $t\le\tau<T$. From $u, u_1\in W_t^q$ we see $u_2\in W_t^q$. It follows that

(3.1) $$\|u_2\|_{W_t^q}=2\|a\|_{\mathcal{T}_t^q}<\infty.$$

So we have $a\in\mathcal{T}_t^q$ and get $F_t^q=\mathcal{T}_t^q$.

From (3.1) we see that
$$2\|a\|_{\mathcal{T}_t^q}=\|u_2\|_{W_t^q}\le\|u\|_{W_t^q}+\|u_1\|_{W_t^q}.$$
By [2] (see (1.2)) it follows
$$\|u_1\|_{W_t^q}\le C\Big(\int_t^T \|\dot u(\tau)+A(t)u(\tau)\|^q d\tau\Big)^{1/q}\le C\|u\|_{W_t^q}.$$
Then we get $2\|a\|_{\mathcal{T}_t^q}\le C\|u\|_{W_t^q}$. This holds for all $u\in W_t^q$ with $u(t)=a$. It follows
$$2\|a\|_{\mathcal{T}_t^q}\le C\inf\{\|u\|_{W_t^q}:\ u\in W_t^q,\ u(t)=a\}=C\|a\|_{F_t^q}.$$

Therefore, we obtain $F_t^q=\mathcal{T}_t^q$ with equivalent norms $\|a\|_{\mathcal{T}_t^q}$, $\|a\|_{F_t^q}$.

The properties ii) and iii) follow immediately.

In the next lemma we shall state the crucial a priori estimate for (1.1).

Lemma 2. *Let* $1<q<\infty$, $u\in W_0^q$ *and set* $f(\tau)=\dot u(\tau)+A(\tau)u(\tau)$ *for* $0\leq\tau<T$ *where* $0<T\leq\infty$. *Then under the assumptions* (2.1) a)–f) *there exists some* $C>0$ *independent of* u *and* T *such that*

$$\int_0^T \|\dot u\|^q d\tau+\int_0^T \|A(\tau)u(\tau)\|^q d\tau\leq C\left(\|u(0)\|_{\mathcal{T}_0^q}^q+\int_0^T \|f(\tau)\|^q d\tau\right).$$

Proof. For simplicity, we carry out the proof only for $T=\infty$. Then the case $T<\infty$ will be clear.

First we consider a subinterval $[0,T_1]$ with $0<T_1<\infty$ and then we proceed to the next interval and so on. T_1 will be fixed later on.

Set $a=u(0)$, $u_0(\tau)=e^{-\tau A(0)}a$ and $u_1=u-u_0$. Then by Lemma 1 we have $a\in\mathcal{T}_0^q$, $u_0\in W_0^q$, and therefore $u_1\in W_0^q$. Using [2] we get

$$\left(\int_0^{T_1}\|\dot u_1\|^q d\tau\right)^{1/q}+\left(\int_0^{T_1}\|A(0)u_1\|^q d\tau\right)^{1/q}\leq C_1\left(\int_0^{T_1}\|\dot u_1+A(0)u_1\|^q d\tau\right)^{1/q}. \tag{3.2}$$

Next we use the continuity of $A(\tau)A(0)^{-1}$ for $\tau\geq 0$ in the operator norm by (2.1) e), and for given $\varepsilon>0$ we can choose T_1 so small that

$$\left(\int_0^{T_1}\|[A(\tau)A(0)^{-1}-I]A(0)u_1\|^q d\tau\right)^{1/q}\leq\varepsilon\left(\int_0^{T_1}\|A(0)u_1\|^q d\tau\right)^{1/q}. \tag{3.3}$$

From $u=u_0+u_1$, we get

$$\left(\int_0^{T_1}\|\dot u\|^q d\tau\right)^{1/q}+\left(\int_0^{T_1}\|A(\tau)u(\tau)\|^q d\tau\right)^{1/q}\leq\left(\int_0^{T_1}\|\dot u_0\|^q d\tau\right)^{1/q}$$
$$+\left(\int_0^{T_1}\|A(\tau)u_0(\tau)\|^q d\tau\right)^{1/q}+\left(\int_0^{T_1}\|\dot u_1\|^q d\tau\right)^{1/q}+\left(\int_0^{T_1}\|A(\tau)u_1(\tau)\|^q d\tau\right)^{1/q}.$$

Using (2.1) d) and $u_0(\tau)=e^{-\tau A(0)}u(0)$

$$\left(\int_0^{T_1}\|\dot u_0\|^q d\tau\right)^{1/q}+\left(\int_0^{T_1}\|A(\tau)u_0(\tau)\|^q d\tau\right)^{1/q}\leq\left(\int_0^{T_1}\|\dot u_0\|^q d\tau\right)^{1/q} \tag{3.4}$$
$$+C_1\left(\int_0^{T_1}\|A(0)u_0(\tau)\|^q d\tau\right)^{1/q}\leq C_2\left(\int_0^{T_1}\|A(0)u_0(\tau)\|^q d\tau\right)^{1/q}=C_3\|u(0)\|_{\mathcal{T}_0^q}.$$

Using (3.2) and (3.3), and choosing $\varepsilon>0$ sufficiently small it holds

$$\left(\int_0^{T_1}\|\dot u_1(\tau)+A(\tau)u_1(\tau)\|^q d\tau\right)^{1/q}$$
$$=\left(\int_0^{T_1}\|\dot u_1(\tau)+A(0)u_1(\tau)+[A(\tau)-A(0)]u_1(\tau)\|^q d\tau\right)^{1/q}$$
$$\geq\left(\int_0^{T_1}\|\dot u_1(\tau)+A(0)u_1(\tau)\|^q d\tau\right)^{1/q}-\left(\int_0^{T_1}\|(A(\tau)-A(0))u_1(\tau)\|^q d\tau\right)^{1/q}$$
$$\geq C_1\left(\int_0^{T_1}\|\dot u_1\|^q d\tau\right)^{1/q}+C_2\left(\int_0^{T_1}\|A(0)u_1(\tau)\|^q a\tau\right)^{1/q}-\varepsilon\left(\int_0^{T_1}\|A(0)u_1(\tau)\|^q d\tau\right)^{1/q}$$
$$\geq C_3\left(\int_0^{T_1}\|\dot u_1\|^q d\tau\right)^{1/q}.$$

We use this value as ε in all steps. We also get

$$\left(\int_0^{T_1}\|A(\tau)u_1(\tau)\|^q d\tau\right)^{1/q}\leq C\left(\int_0^{T_1}\|\dot u_1(\tau)+A(\tau)u_1\|^q d\tau\right)^{1/q}+\left(\int_0^{T_1}\|\dot u_1\|^q d\tau\right)^{1/q}$$
$$\leq C\left(\int_0^{T_1}\|\dot u_1(\tau)+A(\tau)u_1(\tau)\|^q d\tau\right)^{1/q}$$

Combining these two inequalities, we obtain

$$\left(\int_0^{T_1}\|\dot{u}_1\|^q d\tau\right)^{1/q}+\left(\int_0^{T_1}\|A(\tau)u_1(\tau)\|^q d\tau\right)^{1/q}\leq C\left(\int_0^{T_1}\|\dot{u}_1+A(\tau)u_1(\tau)\|^q d\tau\right)^{1/q}$$
$$\leq C\left\{\left(\int_0^{T_1}\|\dot{u}+A(\tau)u(\tau)\|^q d\tau\right)^{1/q}+\left(\int_0^{T_1}\|\dot{u}_0+A(\tau)u_0(t)\|^q d\tau\right)^{1/q}\right\}$$
$$\leq C\left\{\left(\int_0^{T_1}\|\dot{u}+A(\tau)u(\tau)\|^q d\tau\right)^{1/q}+\left(\int_0^{T_1}\|\dot{u}_0\|^q d\tau\right)^{1/q}+\left(\int_0^{T_1}\|A(\tau)u_0(\tau)\|^q d\tau\right)^{1/q}\right\}$$
$$\leq M\left(\int_0^{T_1}\|\dot{u}+A(t)u(\tau)\|^q d\tau\right)^{1/q}+N\|u(0)\|_{\mathcal{F}_0^q}.$$

Here M, N are constants. We used (3.4) in the last inequality. Now we obtain the result for the first interval $[0, T_1]$:

$$\left(\int_0^{T_1}\|\dot{u}\|^q d\tau\right)^{1/q}+\left(\int_0^{T_1}\|A(\tau)u(\tau)\|^q d\tau\right)^{1/q}\tag{3.5}$$
$$\leq M\left(\int_0^{T_1}\|\dot{u}+A(\tau)u(\tau)\|^q d\tau\right)^{1/q}+N\|u(0)\|_{F_0^q}.$$

We choose the next interval $[T_1, T_2]$ in the same way as above. Here we define for $T_1\leq\tau\leq T_2$, $u_1=u-u_0$, $u_0(\tau)=e^{-(t-T_1)A(T_1)}a$ and $a=u(T_1)$. In this case we obtain (3.5) with 0 replaced by T_1 and T_2 replaced by T_1, and so on.

Now we shall show how to choose $T_1, T_2, \cdots, T_k, T_{k+1}=\infty$; let $T_0=0$. We choose first the last point T_k by using (2.1) f). Then $[0, T_k]$ is compact. Hence the continuity by (2.1) e) holds uniformly for all $0\leq\tau\leq t\leq T_k$. So we can choose a finite number of points $T_1, \cdots, T_{k-1}$ for the same given value ε as above. Then we get for $\nu=0, 1, 2, \cdots, k$

$$\left(\int_{T_\nu}^{T_{\nu+1}}\|\dot{u}\|^q d\tau\right)^{1/q}+\left(\int_{T_\nu}^{T_{\nu+1}}\|A(\tau)u(\tau)\|^q d\tau\right)^{1/q}$$
$$\leq M\left(\int_{T_\nu}^{T_{\nu+1}}\|\dot{u}+A(\tau)u(\tau)\|^q d\tau\right)^{1/q}+N\|u(T_\nu)\|_{F_{T_\nu}^q}.$$

This leads to

$$\left(\int_0^{\infty}\|\dot{u}\|^q d\tau\right)^{1/q}+\left(\int_0^{\infty}\|A(\tau)u(\tau)\|^q d\tau\right)^{1/q}\tag{3.6}$$
$$\leq M\left(\int_0^{\infty}\|\dot{u}(\tau)+A(\tau)u(\tau)\|^q d\tau\right)^{1/q}+N\sum_{\nu=0}^{k}\|u(T_\nu)\|_{F_{T_\nu}^q}.$$

In the last step of our proof we show that we may remove the terms $\|u(T_\nu)\|_{F_{T_\nu}^q}$ for $\nu>0$. We argue by contradiction. Suppose we find a sequence $u_\rho\in W_0^q$, $\rho=1, 2, \cdots$, such that $\left(\int_0^{\infty}\|\dot{u}_\rho\|^q d\tau\right)^{1/q}+\left(\int_0^{\infty}\|A(\tau)u_\rho(\tau)\|^q d\tau\right)^{1/q}$ $=1$ for all ρ, and $\left(\int_0^{\infty}\|\dot{u}_\rho+A(\tau)u_\rho\|^q d\tau\right)^{1/q}$ and $\|u_\rho(0)\|_{F_0^q}$ tend to 0 as $\rho\to\infty$. Applying (3.6) to u_ρ, we see that

$$1\leq N\liminf_{\rho\to\infty}\left(\sum_{\nu=1}^{k}\|u_\rho(T_\nu)\|_{F_{T_\nu}^q}\right).\tag{3.7}$$

From (3.5) and (2.1) d), we get the estimate

$$\left(\int_0^{T_1}\|\dot{u}\|^q d\tau\right)^{1/q}+\left(\int_0^{T_1}\|A(T_1)u(\tau)\|^q d\tau\right)^{1/q}$$
$$\leq C\left\{\left(\int_0^{T_1}\|\dot{u}+A(\tau)u(\tau)\|^q d\tau\right)^{1/q}+\|u(0)\|_{F_0^q}\right\}.$$

We have also the next estimate using the definition of W_t^q and $\mathcal{F}_t^q$.

$$\|u(T_1)\|_{F^q_{T_1}} \leq \inf_{u(T_1)=v(T_1)} \left\{ \left(\int_{T_1}^{T} \|\dot{v}(t)\|^q dt \right)^{1/q} + \left(\int_{T_1}^{T} \|A(T_1)v(t)\|^q dt \right)^{1/q} \right\}$$
$$= \inf_{u(T_1)=\tilde{v}(T_1)} \left\{ \left(\int_{T_1}^{2T_1-T} -\|\dot{\tilde{v}}(s)\|^q ds \right)^{1/q} + \left(\int_{T_1}^{2T_1-T} -\|A(T_1)\tilde{v}(s)\|^q ds \right)^{1/q} \right\}$$
$$\leq \left(\int_{2T_1-T}^{T_1} \|\dot{u}(t)\|^q dt \right)^{1/q} + \left(\int_{2T_1-T}^{T_1} \|A(T_1)u(t)\|^q d\tau \right)^{1/q}$$
$$= \left(\int_{0}^{T_1} \|\dot{u}(\tau)\|^q d\tau \right)^{1/q} + \left(\int_{0}^{T_1} \|A(T_1)u(\tau)\|^q d\tau \right)^{1/q}.$$

Here we set $\tilde{v}(s)=v(2T_1-s)$ and $T=2T_1$ in the last part. Replacing u by u_ρ we see from the last two estimates and the assumption of contradiction that

$$\|u_\rho(T_1)\|_{F^q_{T_1}} \longrightarrow 0 \qquad \text{as} \quad \rho \longrightarrow \infty.$$

Repeating the same conclusion to the next interval $[T_1, T_2]$, we see that $\|u_\rho(T_2)\|_{F^q_{T_2}} \to 0$ as $\rho \to \infty$, and so on. It follows $\sum_{\nu=1}^{k} \|u_\rho(T_\nu)\|_{F^q_{T_1}} \to 0$ as $\rho \to \infty$. This fact contradicts the assumption. Lemma 2 is thus proved.

We shall complete this section by showing the existence of a solution u of the evolution equation (1.1) for given $a \in \mathcal{T}^q_0$ and $f \in L^q(0, T; X)$.

The existence of the solution is already clear if $A(\tau) \equiv A(0)$ by [2, Theorem 2.3]. Then we use $\varepsilon > 0$ and T_1 as in the proof above to obtain

$$\|(A(\tau)A(0)^{-1} - I)A(0)v\| \leq \varepsilon \|A(0)v\|$$

for $v \in \mathcal{D}(A(0))$ and $\tau \in [0, T_1]$ by (2.1) e). So we see

$$\|[A(\tau) - A(0))v\| \leq \varepsilon \|A(0)v\| \qquad \text{for all } v \in \mathcal{D}(A).$$

Hence we obtain the existence of the solution in the general case $A(\tau)$ by using Kato's perturbation theorem. Then we extend this solution to the next interval $[T_1, T_2]$ and so on. This yields the result of the theorem.

References

[1] G. Dore and A. Venni: On the closedness of the sum of two closed operators. Math. Z., **196**, 189–201 (1987).

[2] Y. Giga and H. Sohr: Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains. J. Func. Anal. (to appear).

[3] J. Prüss and H. Sohr: On operators with bounded imaginary powers in Banach spaces. Math. Z., **203**, 429–452 (1990).

[4] H. Tanabe: Equation of Evolution. Tokyo, Iwanami (1975) (in Japanese); English translation, London, Pitman (1979).

[5] A. Yagi: On the abstract linear evolution equations in Banach spaces. J. Math. Soc. Japan, **28**, 290–303 (1976).

[6] ——: On the abstract evolution equation of parabolic type. Osaka J. Math., **14**, 557–568 (1977).