# 49. The Initial Boundary Value Problems for Linear Symmetric Hyperbolic Systems with Characteristic Boundary

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**1. Introduction.** This paper studies the existence and differentiability of local solutions in time of the mixed initial boundary value problems for first order symmetric hyperbolic systems. We assume that the boundary is characteristic of constant multiplicity. A general theory for the case where the boundary is non-characteristic was developed by Friedrichs [2], Lax-Phillips [3] and Rauch-Massey III [7]. The case of the characteristic boundary has been treated by Lax-Phillips [3], Tsuji [9], Majda-Osher [5] and Rauch [6]. In particular, the tangential regularity of solutions for the latter case was obtained in [6].

The basic estimate we seek to establish is motivated by the work of Yanagisawa-Matsumura [10]. The norm used in that paper seems to be most suitable for our problem in the sense that "the loss of derivatives in the normal directions" is appropriately taken into account. (See also Chen [1].) Although we confine ourselves to the linear theory in this note, the main theorem is formulated in such a way that the applications to the quasilinear initial boundary value problems are possible.

Let $\Omega\subset \boldsymbol{R}^n$ be an open bounded set lying on one side of its boundary $\Gamma$. We treat differential operators of the form

$$L=A_0(v)\partial_t+\sum_{j=1}^{n} A_j(v)\partial_j+B(v),$$

where $\partial_t=\partial/\partial t$, $\partial_j=\partial/\partial x_j$, and $v=(v_1, v_2, \cdots, v_l)^t$ is a given smooth function of the time $t$ and the space variable $x=(x_1, x_2, \cdots, x_n)$. It is assumed that $A_0(\cdot)$, $A_j(\cdot)$, and $B(\cdot)$ are depending smoothly on their arguments. Therefore $A_0(v)$, $A_j(v)$, and $B(v)$ are smoothly varying real $l\times l$ matrices defined for $(t, x)\in[0, T]\times\bar{\Omega}$. We study the mixed initial boundary value problem

(1.1) $\quad Lu=F \quad$ in $[0, T]\times\Omega$,

(1.2) $\quad M(x)u=0 \quad$ in $[0, T]\times\Gamma$,

(1.3) $\quad u(0, x)=f(x) \quad$ for $x\in\Omega$.

The unknown function $u(t, x)$ is a vector-valued function with $l$ components. $M(x)$ is an $l\times l$ real matrix depending smoothly on $x\in\Gamma$. $M$ is of constant rank everywhere. The inhomogeneous term $F$ of the equation and the initial data $f$ are given vector-valued functions defined on $[0, T]\times\bar{\Omega}$ and $\bar{\Omega}$, respectively. Let $\nu=(\nu_1, \nu_2, \cdots, \nu_n)$ be the unit outward normal to

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$\Gamma$. The boundary matrix, $A_\nu(v)=\sum_{j=1}^n A_j(v)\nu_j$, is assumed to be of constant rank on $\Gamma$. We write $u_t=\partial_t u$, $u_{x_j}=\partial_j u$ in the following.

**2. Preliminaries.** Let $H^m(\Omega)$ be the usual Sobolev space. A vector field $\Lambda \in C^\infty(\bar{\Omega}; \boldsymbol{C}^n)$ is tangential iff $\langle \Lambda(x), \nu(x)\rangle=0$ for all $x \in \Gamma$. When $\Omega$ is a bounded open set with smooth boundary, the function space $H_*^m(\Omega)$ is defined as the set of functions satisfying the following properties:

i) Let $\Omega_\delta$ be the set of $x$ in $\Omega$ such that $\mathrm{dist}(x, \Gamma)>\delta$. Then $u \in H^m(\Omega_\delta)$ for $\delta>0$ small enough.

ii) Let $\Lambda_1, \Lambda_2, \cdots, \Lambda_j$ be tangential vector fields and let $\Lambda_1', \Lambda_2', \cdots, \Lambda_k'$ be non-tangential vector fields. Then $\Lambda_1\Lambda_2\cdots\Lambda_j\Lambda_1'\Lambda_2'\cdots\Lambda_k' u \in L^2(\Omega)$, if $j+2k \leq m$.

$H_*^m(\Omega)$ is normed as follows. We choose as usual the covering of $\Gamma$, diffeomorphisms, and cut off functions, say, $\mathcal{O}_j$, $\tau_j$, $\chi_j$, $1\leq j\leq N$. Then $u^{(j)}=(\chi_j u)\circ\tau_j^{-1}$ has as its natural domain $\mathcal{B}_+=\{x \mid |x|<1, x_1>0\}$ with $\Gamma$ corresponding to $x_1=0$. The linearly independent tangential vector fields are given in local coordinates by $\partial_k$, $k=2, \cdots, n$, and the normal vector field $\partial_\nu$ corresponds to $\partial_1$. Let $\chi_0$ be a cut off function such that $\chi_0=0$ on a neighborhood of $\Gamma$ and $\chi_0=1$ on some $\Omega_\delta$. We may assume that $\sum_{j=0}^N \chi_j^2=1$ on $\bar{\Omega}$. Then the norm in $H_*^m(\Omega)$ is

$$\|u\|^2_{H_*^m(\Omega)}=\|\chi_0 u\|^2_{H^m(\Omega)}+\sum_{j=1}^N \|\chi_j u\|^2_{H_*^m(\Omega)}, \tag{2.1}$$

$$\|\chi_j u\|^2_{H_*^m(\Omega)}=\sum_{|\beta|+2k\leq m} \|\partial_*^\beta \partial_1^k u^{(j)}\|^2_{L^2(\mathcal{B}_+)}, \tag{2.2}$$

where $\beta=(\beta_1, \beta_2, \cdots, \beta_n)$, $|\beta|=\beta_1+\beta_2+\cdots+\beta_n$, and $\partial_*^\beta=(x_1\partial_1)^{\beta_1}\partial_2^{\beta_2}\cdots\partial_n^{\beta_n}$. The norms arising from different choices of $\mathcal{O}_j$, $\tau_j$, $\chi_j$ are equivalent.

If $X$ is a Banach space and $I\subset \boldsymbol{R}$ is an interval, then $C^j(I; X)$ is the space of $j$ times continuously differentiable functions on $I$ taking values in $X$. Similary, $C^{j,w}(I; X)$ denotes the space of $j$ times continuously differentiable functions with values in $X$ in the weak topology. First, let

$$X_*^m([0, T]; \Omega)\equiv\bigcap_{j=0}^m C^j([0, T]; H_*^{m-j}(\Omega)).$$

The norm is

$$|||u|||_{X_*^m([0,T];\Omega)}=\sup_{0\leq t\leq T}|||u(t)|||_{m,*}, \tag{2.3}$$

$$|||u(t)|||^2_{m,*}=\sum_{j=0}^m \|\partial_t^j u(t)\|^2_{m-j,*}. \tag{2.4}$$

Note that, if we write $\alpha=(j, \beta)$, $|\alpha|=j+|\beta|$, and $\partial_\star^\alpha=\partial_t^j\partial_*^\beta$, then $|||u(t)|||^2_{m,*}=\sum_{|\alpha|+2k\leq m}\|\partial_\star^\alpha\partial_1^k u(t)\|^2$. $X^m([0, T]; \Omega)$ is defined similarly by using $H^{m-j}(\Omega)$ in place of $H_*^{m-j}(\Omega)$. We obtain $X_*^{m,w}([0, T]; \Omega)$ replacing $C^j$ by $C^{j,w}$ in the definition of $X_*^m([0, T]; \Omega)$.

**3. The existence and differentiability theorem.** We give the main result of this paper.

**Theorem 3.1.** *Let $m\geq 1$ be an integer and let $s=\max(m, 2[n/2]+6)$. The mixed initial boundary value problem* (1.1), (1.2), (1.3) *has a unique solution in $X_*^m([0, T]; \Omega)$, provided the following conditions are satisfied:*

i) *$\Omega \subset \mathbf{R}^n$ is a bounded open set with $C^{s+1}$ boundary $\Gamma$.*
ii) *$M(x)$ is of $C^{s+1}$ class and* dim ker *$M(x)$ is constant on $\Gamma$.*
iii) *$v \in X^s_*([0, T]; \Omega)$ and $\partial_t^i v(0) \in H^{m-i}(\Omega)$, $0 \le i \le m-1$.*
iv) *$v(t, x) \in$* ker *$M(x)$ for $(t, x) \in [0, T] \times \Gamma$.*
v) *$A_j(v)$, $j=0, 1, \cdots, n$, are real symmetric matrices for $(t, x) \in [0, T] \times \bar{\Omega}$, if $v$ takes values in $\mathbf{R}^l$ and satisfies* iii), iv). *In addition, $A_0(v)$ is positive definite on $[0, T] \times \bar{\Omega}$.*
vi) *There exists a linear subspace $\mathcal{N}(x)$ of $\mathbf{C}^l$ such that, for any $v$ as in* v), *we have* ker *$A_{\nu(x)}(v(t, x)) = \mathcal{N}(x)$, $(t, x) \in [0, T] \times \Gamma$.*
vii) dim *$\mathcal{N}(x)$ is constant on $\Gamma$.*
viii) ker *$M(x)$ is maximally nonnegative for $A_\nu(v)$, where $v$ is as in* v).
ix) *$F \in X^m_*([0, T]; \Omega)$, $\partial_t^i F(0) \in H^{m-i-1}(\Omega)$, $0 \le i \le m-1$, and $f \in H^m(\Omega)$.*
x) *The compatibility conditions up to order $m-1$ are satisfied, i.e., $M$ "$\partial_t^i u(0)$"$=0$ on $\Gamma$, $0 \le i \le m-1$.*

*The solution obeys the estimate*

$$|||u(t)|||_{m,*} \le C(M_{s-1})(\|f\|_m + |||F(0)|||_{m-1}) e^{C(M_s)t} + \int_0^t e^{C(M_s)(t-\tau)} |||F(\tau)|||_{m,*} d\tau, \tag{3.1}$$

*for $t \in [0, T]$, where $M_{s-1}$ and $M_s$ are constants such that $|||v|||_{X^r_*([0,T];\Omega)} \le M_r$ for $r = s-1, s$. $C(M_{s-1})$ and $C(M_s)$ denote the constants depending only on $M_{s-1}$ and $M_s$, respectively.*

**Remark 1.** Given the system (1.1) and the initial data (1.2), "$\partial_t^i u(0)$" is defined by formally taking $i-1$ time derivatives of the system, solving for $\partial_t^i u$ and evaluating at time $t=0$.

**Remark 2.** The conditions for the set of data $f, F$ are somewhat stringent. But, by a limit argument, we can obtain the general conditions for $f, F$ leading to solutions in $X^m_*([0, T]; \Omega)$. This will be discussed elsewhere.

**4. Outline of proof.** First we show the existence of approximate systems and approximate initial data.

**Lemma 4.1.** *Let $f, F$, and $v$ be as in Theorem* 3.1. *Then there exist sequences $f_k, F_k$, and $v_k$, satisfying the following properties:*
i) *$f_k \in H^{m+2}(\Omega)$, $k \ge 1$, and $f_k \longrightarrow f$ in $H^m(\Omega)$.*
ii) *$F_k \in H^{m+2}([0, T] \times \Omega)$, $k \ge 1$, and $F_k \longrightarrow F$ in $X^m_*([0, T]; \Omega)$. Furthermore, $\partial_t^i F_k(0) \longrightarrow \partial_t^i F(0)$ in $H^{m-1-i}(\Omega)$ for $0 \le i \le m-1$.*
iii) *$v_k \in X^{s+1}([0, T]; \Omega)$, $k \ge 1$, and $v_k \longrightarrow v$ in $X^s_*([0, T]; \Omega)$. In addition, $v_k(t, x) \in$* ker *$M(x)$ for $(t, x) \in [0, T] \times \Gamma$.*
iv) *$M$"$\partial_t^i u^k(0)$"$=0$ on $\Gamma$ for $0 \le i \le m$, where "$\partial_t^i u^k(0)$" is defined like "$\partial_t^i u(0)$" but using $f_k$, $F_k$, and $v_k$ in place of $f, F$, and $v$.* (*See Remark* 1.)

This lemma can be proved by combining the arguments in [7], [8] with the trace theorem in $H^m_*(\Omega)$. Next we apply the method, which is usually called the non-characteristic regularization, to the approximate systems. (See, e.g., [6], [8].) Let us consider the following mixed problems.

$$(4.1)\qquad A_0(v_k)u_t^{(k)}+\sum_{j=1}^{n}A_j(v_k)u_{x_j}^{(k)}+B(v_k)u^{(k)}+\frac{1}{C(k)^2}\sum_{j=1}^{n}\nu_j u_{x_j}^{(k)}$$

$$=F_k+\frac{1}{C(k)^2}\sum_{j=1}^{n}\nu_j U_{x_j}^k \qquad \text{in } [0,T]\times\Omega,$$

$$(4.2)\qquad Mu^{(k)}=0 \qquad \text{in } [0,T]\times\Gamma,$$

$$(4.3)\qquad u^{(k)}(0,x)=f_k(x) \qquad \text{for } x\in\Omega.$$

Here $U^k\in X^{m+2}([0,T];\Omega)$ satisfies $\partial_t^i U^k(0)=$"$\partial_t^i u^k(0)$", $0\le i\le m$, and $\partial_t^{m+1}U^k(0)=0$. $C(k)$ denotes a constant such that $|||U^k|||_{X^{m+2}([0,T];\Omega)}\le C(k)$.

Notice that, (i) the boundary $\Gamma$ is non-characteristic for the system (4.1), (ii) the boundary condition (4.2) is still maximally nonnegative on $[0,T]\times\Gamma$ for the system (4.1), (iii) the compatibility conditions up to order $m$ are satisfied for the mixed problem (4.1), (4.2), (4.3). These observations lead to the following

**Proposition 4.2.** *The mixed initial boundary value problem* (4.1), (4.2), (4.3) *has a unique solution* $u^{(k)}$ *in* $X^{m+1}([0,T];\Omega)$, *which obeys the estimate*

$$(4.4)\qquad |||u^{(k)}(t)|||_{m,*}\le C(M_{s-1})$$
$$\times\left\{|||u^{(k)}(0)|||_{m,*}+|||F_k(0)|||_{m-1,*}+\frac{1}{C(k)^2}(1+T)|||U^k|||_{X^{m+2}([0,T];\Omega)}\right\}e^{C(M_s)t}$$
$$+\int_0^t e^{C(M_s)(t-\tau)}|||F_k(\tau)|||_{m,*}d\tau,$$

*for* $t\in[0,T]$. *Here the constants* $C(M_{s-1})$ *and* $C(M_s)$ *are as in Theorem* 3.1.

As for the proof of this proposition, the existence of solution is shown by applying Theorem A.1 in [8]. The proof of the estimate (4.4) will be sketched in the next section. The uniform estimates for $u^{(k)}$ which we seek for follows from Proposition 4.2. Then by a standard argument, we obtain

**Proposition 4.3.** *The mixed initial boundary value problem* (1.1), (1.2), (1.3) *has a unique solution* $u$ *in* $X_*^{m,w}([0,T];\Omega)$.

Using Rauch's mollifier given in [6] and following the arguments in [4], we have finally

**Proposition 4.4.** *The solution* $u$ *obtained in Proposition* 4.3 *lies in* $X_*^m([0,T];\Omega)$.

Thus the proof of the first part of Theorem 3.1 is completed. The estimate (3.1) is obtained by letting $k\to\infty$ in (4.4).

**5. Proof of the estimate (4.4).** Using localization and changes of the dependent variables, we reduce to the case where both $\ker M(x)$ and $\ker A_\nu(v)\equiv\mathcal{N}(x)$ are independent of $x$. This is realized by constructing a smooth unitary matrix-valued function defined on each of the patchs. The relevant potion of the boundary $\Gamma$ is mapped to the hyperplane $x_1=0$ by changes of the independent $x$ variables, and the support of $u$ is contained in $\{x\,|\,|x|<1,\ x_1\ge0\}$. Thus we have the transformed mixed problem

$$(5.1)\qquad \tilde{A}_0(v)u_t+\sum_{j=1}^{n}\tilde{A}_j(v)u_{x_j}+\tilde{B}(v)u-\varepsilon u_{x_1}=\tilde{F}_\varepsilon \qquad \text{in } [0,T]\times\{x\,|\,x_1>0\},$$

(5.2) $\tilde{M}u=0$ on $[0,T]\times\{x\mid x_1=0\}$,

(5.3) $u(0,x)=\tilde{f}(x)$ for $x\in\{x\mid x_1\geq 0\}$,

where $\varepsilon>0$ is a small parameter and $\tilde{M}$ is a constant matrix. We write, for $0\leq j\leq n$,

$$\tilde{A}_j=\begin{pmatrix}\tilde{A}_j^{I\,I} & \tilde{A}_j^{I\,II}\\ \tilde{A}_j^{II\,I} & \tilde{A}_j^{II\,II}\end{pmatrix},\qquad \tilde{B}=\begin{pmatrix}\tilde{B}^{I\,I} & \tilde{B}^{I\,II}\\ \tilde{B}^{II\,I} & \tilde{B}^{II\,II}\end{pmatrix}.$$

Here $\tilde{A}_j^{I\,I}$, $\tilde{B}^{I\,I}$ are $p\times p$ matrices and $\tilde{A}_j^{II\,II}$, $\tilde{B}^{II\,II}$ are $q\times q$ matrices. We have $l=p+q$ and $q=\dim\mathcal{N}(x)$. Since $-\tilde{A}_1(v)$ is the boundary matrix of the system (5.1), it follows from conditions vi), vii) of Theorem 3.1 that $\tilde{A}_1^{I\,I}$ is a $p\times p$ invertible matrix on $[0,T]\times\{x\mid x_1\geq 0\}$. We have also that $\tilde{A}_1^{II\,I}$, $\tilde{A}_1^{I\,II}$, and $\tilde{A}_1^{II\,II}$ vanish on $[0,T]\times\{x\mid x_1=0\}$. We write $u=(u^I,u^{II})^t$, where $u^I\in\boldsymbol{C}^p$, $u^{II}\in\boldsymbol{C}^q$. Similarly, $\tilde{F}_\varepsilon=(\tilde{F}_\varepsilon^I,\tilde{F}_\varepsilon^{II})^t$. The observations preceding Proposition 4.2 are valid for the mixed problem (5.1), (5.2), (5.3). The estimate (4.4) follows from Lemma 5.1 given below and the Gronwall's inequality. We write $|||\cdot|||_{m,tan}^2=\sum_{|\alpha|\leq m}\|\partial_\star^\alpha\cdot\|^2$ and $|||\cdot|||_{m,(*)}^2=\sum_{|\alpha|+2k\leq m,\,k\geq 1}\|\partial_\star^\alpha\partial_1^k\cdot\|^2$ in the following. Note that $|||\cdot|||_{m,tan}^2+|||\cdot|||_{m,(*)}^2=|||\cdot|||_{m,*}^2$.

**Lemma 5.1.** *Let $u$ satisfy* (5.1), (5.2), (5.3). *Then, for $t\in[0,T]$,*

(5.4) $$|||u(t)|||_{m,tan}\leq C|||u(0)|||_{m,tan}+C(M_s)\int_0^t(|||u(\tau)|||_{m,*}+|||u^I(\tau)|||_{m+1,(*)}+|||\tilde{F}_\varepsilon(\tau)|||_{m,tan})d\tau,$$

(5.5) $$|||u(t)|||_{m-1,*}\leq C|||u(0)|||_{m-1,*}+C(M_{s-1})\int_0^t(|||u(\tau)|||_{m,*}+|||u^I(\tau)|||_{m+1,(*)}+|||\tilde{F}_\varepsilon(\tau)|||_{m-1,*})d\tau,$$

(5.6) $$|||u^I(t)|||_{m+1,(*)}\leq C(M_{s-1})\times(|||u(t)|||_{m-1,*}+|||u(t)|||_{m,tan}+|||u^{II}(t)|||_{m,(*)}+|||\tilde{F}_\varepsilon(t)|||_{m-1,*}),$$

(5.7) $$|||u^{II}(t)|||_{m,(*)}\leq C|||u^{II}(0)|||_{m,(*)}+C(M_s)\int_0^t(|||u^I(\tau)|||_{m+1,(*)}+|||u(\tau)|||_{m,*}+|||\tilde{F}_\varepsilon(\tau)|||_{m,(*)})d\tau,$$

*where $C$ is a constant independent of $\varepsilon$.*

In the proof of (5.4), the main terms to be estimated are the commutator parts $[\partial_\star^\alpha,\tilde{A}_1]u_{x_1}$, $0<|\alpha|\leq m$, which include the terms of the form $(\partial_\star^\gamma Z)\partial_\star^{\alpha-\gamma}\partial_1 w$, $|\gamma|=1$. Here $Z$ is $\tilde{A}_1^{I\,II}$, $\tilde{A}_1^{II\,I}$, or $\tilde{A}_1^{II\,II}$ and $w$ is $u^I$ or $u^{II}$. We utilize the fact that $(\partial_\star^\gamma Z)\partial_\star^{\alpha-\gamma}\partial_1$ is a tangential vector field, since $Z$ vanishes on $[0,T]\times\{x\mid x_1=0\}$. The estimate (5.6) is obtained by expressing $\partial_1 u^I$ in terms of $\partial_1 u^{II}$ and the tangential derivatives of $u$, and then using the equation (5.1) and the fact that $\tilde{A}_1^{I\,I}$ is invertible. To show (5.7), we rewrite (5.1) as

$$\tilde{A}_0^{II\,II}u_t^{II}+\sum_{j=1}^n\tilde{A}_j^{II\,II}u_{x_j}^{II}-\varepsilon u_{x_1}^{II}=-(\tilde{A}_0^{II\,I}u_t^I+\sum_{j=1}^n\tilde{A}_j^{II\,I}u_{x_j}^I+\tilde{B}^{II\,I}u^I+\tilde{B}^{II\,II}u^{II})+\tilde{F}_\varepsilon^{II}\quad\text{in }[0,T]\times\{x\mid x_1>0\},$$

and make use of the fact that $\tilde{A}_1^{II\,II}$ vanishes on $[0,T]\times\{x\mid x_1=0\}$.

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