47. On the Cauchy-Kowalevskaya Theorem for Systems[†]

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§1. Introduction and result. In this note, we shall study the Cauchy-Kowalevskaya theorem for systems of partial differential equations which are nondegenerate on ∂_t. (We say that a system is "nondegenerate on ∂_t" when it satisfies the major premise in Theorem 3.1 in M. Miyake [5].) Let Ω be an open set in $C_t^1 \times C_x^l$. If we require the Cauchy problem is uniquely solvable in $C[[t, x]]$, by virtue of Theorem 3.1 in M. Miyake [5], it is enough to consider the following Cauchy problem:

$$(1.1)\qquad \begin{cases} \partial_t u_i(t,x) - \sum_{j=1}^{N} a_{ij}(t,x,\partial_x)u_j(t,x) = f_i(t,x), \\ u_i(t_\circ, x) = \varphi_i(x), \qquad (1 \le i \le N), \end{cases}$$

where all coefficients are holomorphic in Ω, f_i $(1 \le i \le N)$ are given holomorphic functions, φ_i $(1 \le i \le N)$ are holomorphic initial data and u_i $(1 \le i \le N)$ are unknown functions.

We also denote (1.1) by

$$(1.1')\qquad \begin{cases} P(t,x,D_t,D_x)u \equiv D_t u(t,x) - A(t,x,D_x)u(t,x) = f(t,x), \\ \qquad u(t_\circ, x) = \varphi(x), \end{cases}$$

where D_t and D_x are $(\sqrt{-1})^{-1}\partial_t$ and $(\sqrt{-1})^{-1}\partial_x$ respectively, $A(t,x,D_x)$ is an $N \times N$ matrix and u, f and φ are N-vectors.

We say that the Cauchy-Kowalevskaya theorem holds for $P(t,x,D_t,D_x)$ when, for any $(t_\circ, x_\circ)$ in Ω, any neighborhood ω of $(t_\circ, x_\circ)$, any $f(t,x)$ in $\mathcal{H}(\omega)$ and any $\varphi(x)$ in $\mathcal{H}(\omega \cap \{t=t_\circ\})$, there exists an unique holomorphic solution $u(t,x)$ of (1.1′) in a neighborhood of $(t_\circ, x_\circ)$. Here, $f(t,x) \in \mathcal{H}(\omega)$ means that $f(t,x)$ is holomorphic in ω, and so on.

When the order of $A(t,x,D_x)$ is at most one, the Cauchy-Kowalevskaya theorem holds. So, we are interested in the case that the order ν of $A(t,x,D_x)$ is greater than one.

In the case of constant coefficients, the necessary and sufficient condition for the Cauchy-Kowalevskaya theorem is that the characteristic polynomial $\det(\tau I - A(t,x,\zeta))$ is a Kowalevskian polynomial, that is, its degree on τ and ζ is at most N. (See S. Mizohata [6].) As was clarified in [6], the above condition is neither necessary nor sufficient in the case of variable coefficients. In [6], S. Mizohata proposed a necessary condition. Following it, M. Miyake obtained the necessary and sufficient condition in case

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of $l=1$ in [3]. He also proposed a necessary condition in the general case through a construction of a formal solution. (See also [4].)

Our result is a generalization of M. Miyake's one. However, we do not adopt the idea of Volevich system but we introduce weighting operators. (See W. Matsumoto [2].) Therefore, we consider the order of each entry of matrix of differential operator as the normal one. Our standing point is the normal form of matrix in the class of meromorphic formal symbols, which is developed in W. Matsumoto [2]. We recall it.

Definition. We say that a formal sum $\sum_{i=0}^{\infty} p_i(t, x, \zeta)$ is a meromorphic symbol of order ν when it satisfies the following conditions.

1) There exists a conic analytic set Σ in $\Omega \times (\boldsymbol{C}_\zeta^l \backslash \{O\})$ such that $p_i(t, x, \zeta)$ belongs to $\mathcal{M}(\Omega \times (\boldsymbol{C}_\zeta^l \backslash \{O\})) \cap \mathcal{H}(\Omega \times (\boldsymbol{C}_\zeta^l \backslash \Sigma))$ and it is homogeneous of degree $\nu - i$.
2) When $\omega_\circ \times \Gamma_\circ$ in $\Omega \times (\boldsymbol{C}_\zeta^l \backslash \{O\})$ is conic and $(\omega_\circ \times \Gamma_\circ) \cap \{\|\zeta\|=1\}$ is compact in $(\Omega \times (\boldsymbol{C}_\zeta^l \backslash \Sigma)) \cap \{\|\zeta\|=1\}$, there exist positive constants C and R for which $p_i(t, x, \zeta)$ satisfies the following estimate on $\omega_\circ \times \Gamma_\circ$:

$$(1.2) \qquad |p_{i(\alpha)}^{(\beta)}(t, x, \zeta)| \leq C R^{i+|\alpha|+|\beta|} i! |\alpha|! |\beta|! \|\zeta\|^{\nu-i-|\beta|}, \quad (i \in \boldsymbol{Z}_+, \alpha \in \boldsymbol{Z}_+^{l+1}, \beta \in \boldsymbol{Z}_+^l).$$

Here, we use the usual notation: $p_{(\alpha)}^{(\beta)}(t, x, \zeta) = D_{t,x}^\alpha \partial_\zeta^\beta p(t, x, \zeta)$ ($\alpha \in \boldsymbol{Z}_+^{l+1}$ and $\beta \in \boldsymbol{Z}_+^l$) and $\|\zeta\|^2 = |\mathrm{Re}\, \zeta|^2 + |\mathrm{Im}\, \zeta|^2$.

Remark. For α and β in $\boldsymbol{Z}_+^l$, we also denote $D_x^\alpha \partial_\zeta^\beta p(t, x, \zeta)$ by $p_{(\alpha)}^{(\beta)}(t, x, \zeta)$.

We denote the set of the meromorphic formal symbols by $\mathcal{S}_M(\Omega)$. It is an algebra with the product: $(p \circ q) = r$ where $p = \sum_{i=0}^{\infty} p_i$, $q = \sum_{i=0}^{\infty} q_i$, $r = \sum_{i=0}^{\infty} r_i$ and $r_i = \sum_{j+k+|\alpha|=i} (1/\alpha!) p_j^{(\alpha)} q_{k(\alpha)}$.

In [2], W. Matsumoto gave a normal form of matrix with entries in $\mathcal{S}_M(\Omega)$.

Theorem 0 (=Theorem 3.1 in [2]). *Let* $\mathrm{A}(t, x, \zeta)$ *be an* $N \times N$ *matrix with entries in* $\mathcal{S}_M(\Omega)$ *of order* ν *and* $P(t, x, D_t, \zeta)$ *be* $D_t - \mathrm{A}(t, x, \zeta)$. *We suppose that the principal part of* $\mathrm{A}(t, x, \zeta)$ *is holomorphic in* $\Omega \times (\boldsymbol{C}_\zeta^l \backslash \{O\})$ *and has eigenvalue* $\lambda_j(t, x, \zeta)$ *with constant multiplicity* m_j $(1 \leq j \leq d$, $\sum_{j=1}^d m_j = N)$. *Then, there exist* $\{r_j\}_{1 \leq j \leq d}$, $\{n_{jk}\}_{1 \leq j \leq d, 1 \leq k \leq r_j}$ $(\sum_{k=1}^{r_j} n_{jk} = m_j)$ *and an invertible matrix* $\mathcal{N}(t, x, \zeta)$ *in* $\mathcal{S}_M(\Omega)$ *such that*

$$(1.3) \qquad \begin{gathered} \mathcal{N}^{-1} \circ P \circ \mathcal{N} = \oplus_{1 \leq j \leq d} \oplus_{1 \leq k \leq r_j} Q^{jk}(t, x, D_t, \zeta), \\ Q^{jk}(t, x, D_t, \zeta) = I_{n_{jk}}(D_t - \lambda_j(t, x, \zeta)) - \mathcal{D}^{jk}(t, x, \zeta), \\ \mathcal{D}^{jk}(t, x, \zeta) = \sum_{i=0}^{\infty} \mathcal{D}_i^{jk}(t, x, \zeta), \qquad \mathcal{D}_0^{jk} = J_{n_{jk}} \zeta_1^\nu, \\ \mathcal{D}_i^{jk}(t, x, \zeta) = \begin{pmatrix} 0 \\ * \cdots * \end{pmatrix}, \qquad (i \geq 1), \end{gathered}$$

where I_q *is the unit matrix of order* q, J_q *is the Jordan matrix of order* q *with zero eigenvalue and* $\mathcal{D}^{jk}(t, x, \zeta)$ *is an* $n_{jk} \times n_{jk}$ *matrix with entries in* $\mathcal{S}_M(\Omega)$.

Remark. In Theorem 0, we take ζ_1 as a holomorphic scale instead of $\|\zeta\|$. For example, we rewrite $\zeta_1\zeta_2$ as $(\zeta_2/\zeta_1)\zeta_1^2$, where ζ_1^2 indicates the order of the symbol. Of course, we can replace it by any $\zeta_j (2 \leq j \leq l)$.

Remark. If the entries of $A_0(t,x,\zeta)$ are polynomials in ζ and if l is greater than one, d must be one.

Now, we can state our result.

Theorem. *Suppose that the entries of $A(t,x,D_x)$ is partial differential operators of order ν with holomorphic coefficients in Ω. The following (a), (b) and (c) are equivalent.*

(a) *The Cauchy-Kowalevskaya theorem holds for $P(t,x,D_t,D_x)$.*

(b) *The principal symbol of $A(t,x,\zeta)$ is nilpotent in $\Omega\times C^l_\zeta$. In the normal form of $P(t,x,D_t,\zeta)$ in $\mathcal{S}_M(\Omega)$, we set*

$$\sum_{i=1}^{\infty}\mathcal{D}^{jk}_i(t,x,\zeta)=\begin{pmatrix} & 0 & \\ b^{jk}(1), & \cdots, & b^{jk}(n_{jk})\end{pmatrix}.$$

Then, further, the following relations holds:

$$\text{Order of } b^{jk}(q)\le 1-(\nu-1)(n_{jk}-q),\qquad (1.4)$$
$$(1\le j\le d,\ 1\le k\le r_j,\ 1\le q\le n_{jk}).$$

(c) *The system $P(t,x,D_t,\zeta)$ can be reduced to a first order system by a similar transformation in the meromorphic formal symbol class $\mathcal{S}_M(\Omega)$.*

§ 2. Sketch of the proof of theorem.

(a)$\Rightarrow$(b). Considering $P(t,x,D_t,\zeta)$ and its normal form out of the pole set of $\mathcal{N}(t,x,\zeta)$, $\mathcal{N}(t,x,\zeta)^{-1}$ and $\mathcal{D}^{jk}(t,x,\zeta)$, we can prove this assertion by a usual microlocal energy method. (See H. Yamahara [7].)

(b)$\Rightarrow$(c). We apply a similar transformation to the normal form of $P(t,x,D_t,\zeta)$ by a weighting operator

$$\mathcal{W}=\oplus\mathcal{W}^{jk},\qquad \mathcal{W}^{jk}=\mathrm{diag}\,(\zeta_1^{(\nu-1)(n_{jk}-1)},\zeta_1^{(\nu-1)(n_{jk}-2)},\cdots,\zeta_1^{(\nu-1)},1),$$

$\mathcal{W}^{-1}\circ\mathcal{N}^{-1}\circ P\circ\mathcal{N}\circ\mathcal{W}$ becomes a first order system in $\mathcal{S}_M(\Omega)[D_t]$.

Thus, our main purpose of this section is to give an outline of the proof from (c) to (a), that is, we shall show the sufficiency of (c) for the Cauchy-Kowalevskaya theorem for $P(t,x,D_t,D_x)$.

If we can solve (1.1′) with $f(t,x)=0$ for arbitrary $t_\circ$ and if we can obtain a uniform estimate of the solution on $t_\circ$, we can also solve (1.1′) with a general $f(t,x)$ by Duhamel's principle. Then, we only consider (1.1′) in the case of $f(t,x)=0$. The equations become

$$D_tu=A(t,x,D_x)u.\qquad (2.1)$$

This implies that we can express D_t^ju by a linear sum of spacial derivatives of $u(t,x)$. Let us set

$$D_t^ju=A[j](t,x,D_x)u.\qquad (2.2)$$

$A[j](t,x,\zeta)$ is obtained successively by the following formulas:

$$\begin{cases} A[0]=I,\\ A[j+1](t,x,\zeta)=(D_tA[j])(t,x,\zeta)+(A[j]\circ A)(t,x,\zeta).\end{cases}\qquad (2.3)$$

Remark that $A[j](t,x,\zeta)$ is holomorphic in $\Omega\times C^l_\zeta$ and a polynomial on ζ.

Thus, the fundamental solution of (2.1) can be expressed formally in the following way:

$$E(t,x,D_x;t_\circ)=\sum_{j=0}^{\infty}\frac{\{\sqrt{-1}(t-t_\circ)\}^j}{j!}A[j](t_\circ,x,D_x).\qquad (2.4)$$

We set $A[j](t, x, \zeta)=\sum_{i=0}^{\infty} A[j]_i(t, x, \zeta)$. In effect, this is a finite sum. If we can show the following proposition, the righthand side of (2.4) really gives the local fundamental solution acting on the holomorphic functions.

Proposition. *For an arbitrary compact set K in Ω, there exist constants ν_0, C, R and R_0 independent of j, for which the following estimate on $A[j](t, x, \zeta)$ holds on $K\times C^l_\zeta$:*

$$(2.5)\quad |A[j]^{(\beta)}_{i(\alpha)}(t, x, \zeta)|\leq CR_0^j \sum_{h=0}^{j} R^{j-h+i+|\alpha|+|\beta|}(j-h)!\,i!\,|\alpha|!\,|\beta|!\,\|\zeta\|^{\nu_0+h-i-|\beta|},$$
$$(i\in Z_+,\ \alpha\in Z_+^{l+1},\ \beta\in Z_+^l).$$

From now on, we shall show the estimate (2.5). Under Condition (c), there exists an invertible matrix $\mathcal{N}(t, x, \zeta)$ in $\mathcal{S}_M(\Omega)$ such that $\mathcal{N}^{-1}\circ P\circ\mathcal{N}=D_t-\mathcal{B}(t, x, \zeta)$ where $\mathcal{B}(t, x, \zeta)$ belongs to $\mathcal{S}_M(\Omega)$ and its order is at most one. We define $\{B[j](t, x, \zeta)\}_{j=0}^{\infty}$ from $\mathcal{B}(t, x, \zeta)$ by the same way as $\{A[j](t, x, \zeta)\}_{j=0}^{\infty}$, that is, they are determined by

$$(2.6)\quad \begin{cases} B[0]=I, \\ B[j+1](t, x, \zeta)=(D_tB[j])(t, x, \zeta)+(B[j]\circ\mathcal{B})(t, x, \zeta). \end{cases}$$

Let us denote again the union of the pole sets of $\mathcal{N}$, $\mathcal{N}^{-1}$ and $\mathcal{B}$ by Σ. Applying Lemma 1.2 in L. Boutet de Monvel and P. Krée [1], we get the following estimate from (2.6). (See also (1.13) and Proposition 1.2 in W. Matsumoto [2].)

Lemma 1. *When $\omega_0\times\Gamma_0$ is conic and $(\omega_0\times\Gamma_0)\cap\{\|\zeta\|=1\}$ is compact in $(\Omega\times(C^l_\zeta\backslash\Sigma))\cap\{\|\zeta\|=1\}$, the following estimate holds:*

$$(2.7)\quad |B[j]^{(\beta)}_{i(\alpha)}(t, x, \zeta)|\leq CR_0^j \sum_{h=0}^{j} R^{j-h+i+|\alpha|+|\beta|}(j-h)!\,i!\,|\alpha|!\,|\beta|!\,\|\zeta\|^{h-i-|\beta|}.$$

There exists the following relation between $\{A[j](t, x, \zeta)\}_{j=0}^{\infty}$ and $\{B[j](t, x, \zeta)\}_{j=0}^{\infty}$:

$$(2.8)\quad A[j](t, x, \zeta)=\sum_{k=0}^{j}\frac{j!}{k!\,(j-k)!}(D_t^{j-k}\mathcal{N})\circ B[k]\circ\mathcal{N}^{-1}(t, x, \zeta).$$

By the relation (2.8) and Lemma 2.1 in [1], Lemma 1 implies the following.

Lemma 2. *When $\omega_0\times\Gamma_0$ is conic and $(\omega_0\times\Gamma_0)\cap\{\|\zeta\|=1\}$ is compact in $(\Omega\times(C^l_\zeta\backslash\Sigma))\cap\{\|\zeta\|=1\}$, the following estimate holds:*

$$(2.9)\quad |A[j]^{(\beta)}_{i(\alpha)}(t, x, \zeta)|\leqq CR_0^j \sum_{h=0}^{j} R^{j-h+i+|\alpha|+|\beta|}(j-h)!\,i!\,|\alpha|!\,|\beta|!\,\|\zeta\|^{\nu_0+h-i-|\beta|},$$

where ν_0 is (Order of $\mathcal{N}$ + Order of $\mathcal{N}^{-1}$).

By virtue of the maximum principle on the holomorphic function, (2.9) holds on $\Omega\times C^l_\zeta$, that is, we have arrived at the estimate (2.5). The last argument was employed on a formal solution in M. Miyake [3]. We divert it to a formal fundamental solution.

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