32. On Some Discrete Reflection Groups and Congruence Subgroups*)

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In this paper we present some congruence subgroups of the groups of automorphisms of the balls and the classical bounded domains of type IV and announce that they are generated by finitely many reflections. The details and the relation between these groups and the differential equations will be given in the forthcoming paper [3, 4].

1. Let A be a Hermitian form with signature $(n-,1+)$ on an $(n+1)$-dimensional vector space V; the form $(u,v):=A(u,v)$ is supposed to be $\boldsymbol{C}$-linear in v and anti-$\boldsymbol{C}$-linear in u. Let

$$V^+=\{v\in V\mid (v,v)>0\},\quad V^0=\{v\in V\mid (v,v)=0\},\quad V^-=\{v\in V\mid (v,v)<0\}.$$

Notice that $\boldsymbol{D}:=V^+/\boldsymbol{C}^\times$ is isomorphic to the unit ball $\{(z_1,\cdots,z_n)\in\boldsymbol{C}^n\mid |z_1|^2+\cdots+|z_n|^2<1\}$ and that the group $Aut(\boldsymbol{D})$ of automorphisms of $\boldsymbol{D}$ is given by the projectivization of the group $\{g\in GL(V)\mid (gu,gv)=(u,v)\}$. Notice also that $\partial\boldsymbol{D}=V^0/\boldsymbol{C}^\times$. For $\alpha\in V^-$, we define the following transformation R_α called the reflection with respect to a root α by

$$v\mapsto v-2\frac{(\alpha,v)}{(\alpha,\alpha)}\alpha,$$

which keeps the Hermitian form A invariant; in particular, it defines an automorphism of $\boldsymbol{D}$. Notice that R_α pointwisely keeps the subspace $\alpha^\perp=\{v\in V\mid (\alpha,v)=0\}$, which is called the mirror of the reflection. A group generated by reflections is called a reflection group.

Let $n=5$ and fix a basis of the linear space V and the Hermitian form $A=(a_{jk})$ $(1\le j,k\le 6)$ as follows:

$$a_{jj}=-2,\quad a_{jk}=\bar{a}_{kj}=-1+i\ (j<k),\quad \text{where } i=\sqrt{-1}.$$

Let $\boldsymbol{Z}[i]$ be the ring of Gauss integers; the full modular group Γ is defined by

$$\Gamma=\{X\in GL(6,\boldsymbol{Z}[i])\mid X^*AX=A\}.$$

The principal congruence subgroup $\Gamma(1-i)$ with respect to the ideal $(1-i)\subset\boldsymbol{Z}[i]$ is defined by

$$\Gamma(1-i)=\{X\in\Gamma\mid X\equiv I_6 \bmod (1-i)\}.$$

An integral root of norm -2 is a vector $\alpha\in V^-$ whose entries are in $\boldsymbol{Z}[i]$ such that $(\alpha,\alpha)=-2$. By definition, for every integral root α of norm -2, the reflection R_α belongs to $\Gamma(1-i)$.

Theorem 1. (1) *The group $\Gamma(1-i)$ is generated by finitely many reflections with respect to integral roots of norm -2, e.g.*

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*) Dedicated to Professor Bernard Morin on his 60th birthday.

$e_1=(1,0,\cdots,0),\cdots,e_6=(0,\cdots,0,1)$, *and* e_j-e_k $(1\leq j<k\leq 6)$.

(2) *The group* $\Gamma(1-i)$ *is a lattice of* $Aut(D)$; *it has* 35 *cusps.*

(3) *The quotient group* $\Gamma/\Gamma(1-i)$ *is isomorphic to the symmetric group of degree* 8 *acting transitively on the set of* 35 *cusps.*

(4) *Let* $\boldsymbol{D}_{reg}$ *be the subset of* $\boldsymbol{D}$ *consisting of points where the group* $\Gamma(1-i)$ *acts freely. Then we have*

$$\boldsymbol{D}_{reg}=\boldsymbol{D}-\bigcup\{\alpha^{\perp}\cap\boldsymbol{D}\,|\,\alpha:\text{integral of norm }-2\}.$$

(5) *The quotient space* $\boldsymbol{D}_{reg}/\Gamma(1-i)$ *is isomorphic to the configuration space* X_8 *of distinct* 8 *(ordered) points on the complex projective line. The space* X_8 *is defined as the following quotient space, which carries a natural structure of a Zarisky open subset of* $\boldsymbol{C}^5$:

$$GL(2,\boldsymbol{C})\backslash\left\{\begin{pmatrix}x_1\cdots x_8\\ y_1\cdots y_8\end{pmatrix}\,\middle|\,\det\begin{pmatrix}x_j & x_k\\ y_j & y_k\end{pmatrix}\neq 0\ (1\leq j<k\leq 8)\right\}/(\boldsymbol{C}^{\times})^8.$$

(6) *The multi-valued inverse map* $X_8\rightarrow\boldsymbol{D}_{reg}$ *is given by integrals*

$$\int_{c_j}\prod_{k=1}^{8}(x_kt-y_k)^{-1/4}dt \qquad 1\leq j\leq 6,$$

where the c_j *are suitable linearly independent cycles.*

The following lemma concerning the parabolic part of the group $\Gamma(1-i)$ plays an important role in proving the theorem. Let us fix a vector subspace $F\in V^0$ of V corresponding to a Γ-rational boundary point. Let us define the parabolic part $\Gamma P(1-i)$ as the subgroup of $\Gamma(1-i)$ which keeps the line F; let $Z(\Gamma P(1-i))$ be its center. Let π be the projection $V\rightarrow V/F$, and $\pi_*\Gamma P(1-i)$ the induced transformation groups on V/F.

Lemma 1. *We have the following splitting exact sequences*

$$0\rightarrow L\rightarrow\pi_*\Gamma P(1-i)\rightarrow G\rightarrow 0,$$
$$0\rightarrow Z(\Gamma P(1-i))\rightarrow\Gamma P(1-i)\rightarrow\pi_*\Gamma P(1-i)\rightarrow 0,$$

where $L\cong(\boldsymbol{Z}+i\boldsymbol{Z})^4$ *and* $G\cong G(4,2,2)^2$ *are the lattice and the point group of the* 4*-dimensional complex crystallographic group* $\pi_*\Gamma P(1-i)$. *Here* $G(4,2,2)$ *is the imprimitive* 2*-dimensional unitary reflection group of order* 16.

2. Let V_j be a $(j+4)$-dimensional complex vector space; define the integral symmetric matrices as follows:

$$A_j=U\oplus U\oplus(-I_j) \qquad \text{where } U=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix};$$

we write $(u,v)=A_j(u,v)$. Let $\Gamma_{A_j}(2)$ be the principal congruence subgroup

$$\{X\in\Gamma_{A_j}\,|\,X\equiv I_{j+4}\bmod 2\}$$

of the group

$$\Gamma_{A_j}=\{X\in GL(j+4,\boldsymbol{Z})\,|\,{}^tXAX=A\}.$$

For an integral $(j+4)$-vector $\alpha\in V_{j+4}$ such that $(\alpha,\alpha)=-1$, we define the reflection R_α with respect to a root α as above; the reflection R_α belongs to $\Gamma_{A_j}(2)$.

Theorem 2. *For* $j=1,2,3,4$ *and* 5, *the group* Γ_{A_j} *is generated by finite number of reflections with integral roots of norm* -1. *Integral roots with entries* 0 *and* 1 *for* $j=1,2,3,4$ *(with entries* 0, 1 *and* 2 *for* $j=5$*), which are finite in number, give a set of generating reflections.*

Let U_j be $(j+2)$-dimensional complex vector spaces and put $B_j = U \oplus (-I_j)$; we write $(u, v) = B_j(u, v)$. Let $\Gamma_{B_j}(2)$ be the principal congruence subgroup

$$\{X \in \Gamma_{B_j} \mid X \equiv I_{j+2} \bmod 2\}$$

of the group

$$\Gamma_{B_j} = \{X \in GL(j+2, \boldsymbol{Z}) \mid {}^tXBX = B\}.$$

For an integral $(j+2)$-vector $\beta \in U_{j+2}$ such that $(\beta, \beta) = -1$, we define the reflection R_β with respect to a root β as above; the reflection R_β belongs to $\Gamma_{B_j}(2)$. The following lemma is the key to prove Theorem 2; this corresponds to the parabolic part for a 0-dimensional boundary component of the group Γ_{A_j}.

Lemma 2. *For $j=1, 2, 3, 4$ and 5, the group Γ_{B_j} is generated by a finite number of reflections with integral roots of norm -1. The reflections with the following roots $(\alpha_1, \alpha_2; \alpha_3, \cdots, \alpha_{j+2})$ form a set of generators.*

$j=1$: $(0,0;1)$, $(1,0;1)$, $(0,1;1)$,

$j=2$: $(0,0;1,0)$, $(0,0;0,1)$, $(1,0;1,0)$, $(1,0;0,1)$, $(0,1;0,1)$, $(0,1;0,1)$,

$j=3$: $(0,0;1,0,0)$, $(0,0;0,1,0)$, $(0,0;0,0,1)$, $(1,0;1,0,0)$, $(1,0;0,1,0)$, $(1,0;0,0,1)$, $(0,1;1,0,0)$, $(0,1;0,1,0)$, $(0,1;0,0,1)$, $(1,1;\mathbf{1}_3)$,

$j=4$: $(0,0;1,\mathbf{0}_3)$, $\cdots$, $(0,0;\mathbf{0}_3,1)$, $(1,0;1,\mathbf{0}_3)$, $\cdots$, $(1,0;\mathbf{0}_3,1)$, $(0,1;1,\mathbf{0}_3)$, $\cdots$, $(0,1;\mathbf{0}_3,1)$, $(1,1;0,\mathbf{1}_3)$, $\cdots$, $(1,1;\mathbf{1}_3,0)$,

$j=5$: $(0,0;1,\mathbf{0}_4)$, $\cdots$, $(0,0;\mathbf{0}_4,1)$, $(1,0;1,\mathbf{0}_4)$, $\cdots$, $(1,0;\mathbf{0}_4,1)$, $(0,1;1,\mathbf{0}_4)$, $\cdots$, $(0,1;\mathbf{0}_4,1)$, $(1,1;\mathbf{0}_2,\mathbf{1}_3)$, $\cdots$, $(1,1;\mathbf{1}_3,\mathbf{0}_2)$, $(2,1;\mathbf{1}_5)$, $(2,1;\mathbf{1}_5)$,

where $\mathbf{0}_k$ and $\mathbf{1}_k$ stand for the k-tuples of 0's and 1's.

References

[1] K. Matsumoto: On modular functions in 2 variables attached to a family of hyperelliptic curves of genus 3. Ann. Scuola Norm. Sup. di Pisa, **16**, 557–578 (1989).

[2] K. Matsumoto, T. Sasaki and M. Yoshida: The monodromy of the period map of a 4-parameter family of $K3$ surfaces and the hypergeometric function of type (3, 6) (preprint) (1990).

[3] K. Matsumoto and M. Yoshida: Configuration space of 8 points on the projective line and a 5-dimensional Picard modular group (in preparation).

[4] ——: Certain reflection groups acting on bounded symmetric domains of type IV (in preparation).

[5] M. Yoshida: Fuchsian Differential Equations. Wiesbarden, Vieweg Verlag (1987).