31. Minimal Quasi-ideals in Abstract Affine Near-rings. II

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1. Introduction. In ring theory, it is well known that each one of the intersection and the product of a minimal right ideal and a minimal left ideal of a ring is either $\{0\}$ or a minimal quasi-ideal of the ring (see [2]). In [5], this result has been generalized for zero-symmetric near-rings.

The purpose of this note is to extend the above result to a class of abstract affine near-rings. For the basic terminology and notation we refer to [1].

2. Preliminaries. Let N be a near-ring, which always means right one throughout this note.

If A and B are two non-empty subsets of N, then AB denotes the set of all finite sums of the form $\sum a_k b_k$ with $a_k \in A$, $b_k \in B$, and $A*B$ denotes the set of all finite sums of the form $\sum(a_k(a'_k+b_k)-a_k a'_k)$ with $a_k, a'_k \in A$, $b_k \in B$.

A right ideal of N is a normal subgroup R of $(N, +)$ such that $RN \subseteq R$, and a left ideal of N is a normal subgroup L of $(N, +)$ such that $N*L \subseteq L$. A quasi-ideal of N is a subgroup Q of $(N, +)$ such that $N*Q \cap NQ \cap QN \subseteq Q$. Right ideals and left ideals are quasi-ideals. The intersection of a family of quasi-ideals is again a quasi-ideal.

A non-zero quasi-ideal Q of N is minimal if the only quasi-ideal of N contained in Q are $\{0\}$ and Q. Similarly, one defines minimal right ideals and minimal left ideals.

A near-ring N is called an abstract affine near-ring if N is abelian and $N_0 = N_d$, where N_0 and N_d are the zero-symmetric part and the set of all distributive elements of N, respectively.

Let N be an abstract affine near-ring. Then the following hold (see [3] and [4]):

(a) A subgroup L of $(N, +)$ is a left ideal of N if and only if $N_0 L \subseteq L$.

(b) If S is a subgroup of $(N, +)$, then $N_0 S$ is a left ideal of N and SN is a right ideal of N.

(c) A subgroup Q of $(N, +)$ is a quasi-ideal of N if and only if $N_0 Q \cap QN \subseteq Q$.

3. Main results. We start with

Lemma 1. *Let N be an abstract affine near-ring. Then a minimal right (left) ideal of N contained in N_0 is a minimal right (left) ideal of N_0.*

Proof. Let R be a minimal right ideal of N contained in N_0. By [1, Proposition 9.73] we have $R = R_0 + R_c$, where $R_0 = R \cap N_0$ is a right ideal of

N_0, $R_c = R \cap N_c$ where N_c is the constant part of N and $R_0 N_c \subseteq R_c$. Since $R \subseteq N_0$, we have $R = R_0$, $R_c = \{0\}$ and $RN_c = \{0\}$. So, if R' is a non-zero right ideal of N_0 contained in R, then we have $R'N_c = \{0\}$. Therefore we have

$$R'N = R'(N_0 + N_c) = R'N_0 + R'N_c = R'N_0 \subseteq R',$$

whence R' is a right ideal of N. By the minimality of R we have $R' = R$. Thus R is a minimal right ideal of N_0.

Now let L be a minimal left ideal of N contained in N_0. Then L is a left ideal of N_0, since $N_0 L \subseteq L$. On the other hand, if L' is a non-zero left ideal of N_0 contained in L, then we have $N_0 L' \subseteq L'$, that is, L' is a left ideal of N. By the minimality of L we have $L' = L$. Thus L is a minimal left ideal of N_0.

Lemma 2. *Let N be an abstract affine near-ring. Then, for every right ideal R of N and for every non-empty subset A of N, $R * A = R_0 A$ holds, where $R_0 = R \cap N_0$.*

The proof is straightforward and omitted.

Lemma 3. *Let N be an abstract affine near-ring. If Q is a minimal quasi-ideal of N_0, then Q is a minimal quasi-ideal of N.*

Proof. By [3, Corollary] Q is a quasi-ideal of N. On the other hand, let Q' be a non-zero quasi-ideal of N contained in Q, then we have

$$N_0 Q' \cap Q'N_0 \subseteq N_0 Q' \cap Q'N \subseteq Q',$$

whence Q' is a quasi-ideal of N_0. So, by the minimality of Q we have $Q' = Q$. Thus Q is a minimal quasi-ideal of N.

Now we are ready to state the main results of this note.

Theorem 1. *The intersection of a minimal right ideal R and a minimal left ideal L of an abstract affine near-ring N is either $\{0\}$ or a minimal quasi-ideal of N.*

Proof. The intersection $R \cap L = Q$ is a quasi-ideal of N. If $Q \neq \{0\}$, then we assume the existence of a non-zero quasi-ideal Q' such that $Q' \subset Q$.

In case of $N_0 Q' = \{0\}$, Q' would be a left ideal of N such that $\{0\} \subset Q' \subset L$, which contradicts the minimality of L; so we have $N_0 Q' \neq \{0\}$. Since $N_0 Q'$ is a left ideal of N contained in L, by the minimality of L we have $N_0 Q' = L$. Similarly, one can show that $Q'N = R$. Therefore we have $Q = R \cap L = Q'N \cap N_0 Q' \subseteq Q'$, in contradiction with our assumption $Q' \subset Q$. Thus $Q = R \cap L$ $(\neq \{0\})$ is a minimal quasi-ideal of N.

Theorem 2. *The product $R * L$ of a minimal right ideal R and a minimal left ideal L of an abstract affine near-ring N is either $\{0\}$ or a minimal quasi-ideal of N.*

Proof. By Lemma 2, we have $R * L = R_0 L$, where $R_0 = R \cap N_0$.

Suppose that $R_0 L \neq \{0\}$. Since N_c is a right ideal of N, $R_c = R \cap N_c$ is a right ideal of N contained in R. By the minimality of R, either $R_c = \{0\}$ or $R_c = R$. In case of $R_c = R$, we have $R_0 = \{0\}$, because $R = R_0 + R_c$. Hence $R_0 L = \{0\}$, which contradicts our assumption that $R_0 L \neq \{0\}$; so we have $R_c = \{0\}$, $R = R_0$ and $R_0 L = R * L = RL$.

If $N_0(RL) = \{0\}$, then we have

$$N_0(RL) \cap (RL)N = \{0\} \subseteq RL,$$

whence RL is a quasi-ideal of N such that $\{0\} \neq RL = R_0L \subseteq R \cap L$. Moreover, by Theorem 1, $R \cap L$ is a minimal quasi-ideal of N. So, $R * L = RL = R \cap L$ is a minimal quasi-ideal of N.

If $N_0(RL) \neq \{0\}$, $N_0(RL)$ is a non-zero left ideal of N such that $N_0(RL) = N_0(R_0L) \subseteq N_0L \subseteq L$. By the minimality of L, we have $N_0(RL) = L$. Moreover we have

$$RL = R_0L \subseteq R \cap L \subseteq R \subseteq N_0,$$

whence $L = N_0(RL) \subseteq N_0$. Thus R, $L \subseteq N_0$. So, by Lemma 1, R and L are a minimal right ideal and a minimal left ideal of N_0, respectively. Hence, by [5, Theorem 4], $R * L = RL$ is a minimal quasi-ideal of N_0. Therefore, by Lemma 3, $R * L$ is a minimal quasi-ideal of N.

Theorem 3. *The product RL of a minimal right ideal R and a minimal left ideal L of an abstract affine near-ring N is either $\{0\}$ or a minimal quasi-ideal of N.*

Proof. The proof of Theorem 2 shows that $R_c = R \cap N_c$ is either R or $\{0\}$.

In case of $R_c = R$, we have $RL = R$. Let Q be a non-zero quasi-ideal of N contained in R. Since $Q \subseteq R \subseteq N_c$, we have $QN = Q$, that is, Q is a right ideal of N. By the minimality of R, we have $Q = R$. Thus RL is a minimal quasi-ideal of N.

In case of $R_c = \{0\}$, the proof of Theorem 2 shows that $RL = R * L$. So, by Theorem 2, RL is either $\{0\}$ or a minimal quasi-ideal of N.

Now, it is natural to ask whether or not Theorems 1, 2 and 3 hold respectively for arbitrary non-zero-symmetric near-rings. This question is still open.

References

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