29. *The Nonlinear Schrödinger Limit and the Initial Layer of the Zakharov Equations*

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A simple system of equations for the propagation of Langmuir turbulance in an unmagnetized, completely ionized hydrogen plasma was first obtained by Zakharov [6] by means of a two-fluid description of the plasma. In suitably scaled coordinates, the Zakharov equations are the following:

$$(\mathrm{Z}_\lambda)\qquad \begin{aligned} &i\partial_t E_\lambda+\Delta E_\lambda=n_\lambda E_\lambda, \quad t>0, \quad x\in \boldsymbol{R}^N,\\ &\lambda^{-2}\partial_t^2 n_\lambda-\Delta n_\lambda=\Delta|E_\lambda|^2, \quad t>0, \quad x\in\boldsymbol{R}^N,\\ &E_\lambda(0,x)=E_0(x), \quad n_\lambda(0,x)=n_0(x), \quad \partial_t n_\lambda(0,x)=n_1(x), \quad x\in\boldsymbol{R}^N,\end{aligned}$$

where E_λ and n_λ are functions on $\boldsymbol{R}_t\times\boldsymbol{R}_x^N$ with values in $\boldsymbol{C}^N$ and $\boldsymbol{R}$, respectively, $\lambda>1$ is a parameter, and (E_0, n_0, n_1) are given initial data. In these equations E is the slowly varying complex amplitude of the electric field, n is the deviation of the ion density from its equilibrium, and λ is the ion sound speed. In the limit $\lambda\to\infty$ in the second equation of (Z_λ) we formally have the equation $\Delta(n+|E|^2)=0$ so that $n=-|E|^2$ if $n+|E|^2$ vanishes at infinity. Therefore in the limit in (Z_λ) we formally obtain the nonlinear Schrödinger equation

$$(\mathrm{NLS})\qquad \begin{aligned} &i\partial_t E+\Delta E=-|E|^2E, \quad t>0, \quad x\in\boldsymbol{R}^N,\\ &E(0,x)=E_0(x), \quad x\in\boldsymbol{R}^N.\end{aligned}$$

Thus (Z_λ) can be regarded as a natural extension of (NLS), when we take a finite response time of the nonlinear medium of the ion part of the plasma into account, and the limit $\lambda\to\infty$ turns out to be related to an instant response of the medium.

From now on we assume that the initial data (E_0, n_0, n_1) are in the Schwartz space $\mathcal{S}$. Let E be the solution of (NLS) in H^∞ with the maximal existence time $T_{\max}$, where $H^\infty=\bigcap_{k\geq 0}H^k$ and H^k is the usual Sobolev space of order k. If was shown in H. Added and S. Added [1], Ozawa and Tsutsumi [3], Schochet and Weinstein [4], and C. Sulem and P. L. Sulem [5] that if $n_1\in\dot{H}^{-1}$ and $1\leq N\leq 3$, then for any T with $0<T<T_{\max}$ and any $m\in N$ there exist two positive constants C_0 and λ_0 such that for any $\lambda>\lambda_0$ (Z_λ) has a unique solution (E_λ, n_λ) on the interval $[0,T]$ belonging to $C^\infty([0,T]; H^\infty)$ and satisfying

$$\sup_{\lambda\geq\lambda_0}\sup_{0\leq t\leq T}(\|E_\lambda(t)\|_{H^{m+1}}+\|n_\lambda(t)\|_{H^m})\leq C_0,$$

where $\dot{H}^{-1}=\{\psi\in\mathcal{S}'; (-\Delta)^{-1/2}\psi\in L^2\}$, and in particular, $T_{\max}=\infty$ if $N=1$. Moreover, it is shown in [1] that if $n_1=\nabla\cdot\phi$ with $\phi\in\mathcal{S}$, then there exist two

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positive constants C_1 and λ_1 such that for any $\lambda \geq \lambda_1$

$$(1) \qquad \sup_{0\leq t\leq T} (\|E_\lambda(t)-E(t)\|_{H^m}+\|n_\lambda(t)+|E_\lambda(t)|^2-Q_1^{(1)}(\lambda t)\|_{H^{m-1}})$$
$$\leq \begin{cases} C_1\lambda^{-1/2} & \text{if } n_0+|E_0|^2\neq 0 \text{ and } N=1,2,\\ C_1\lambda^{-1/2}\log\lambda & \text{if } n_0+|E_0|^2\neq 0 \text{ and } N=3,\\ C_1\lambda^{-1} & \text{if } n_0+|E_0|^2=0 \text{ and } 1\leq N\leq 3,\end{cases}$$

where $Q^{(1)}(\lambda t)=(\cos\lambda t(-\Delta)^{1/2})(n_0+|E_0|^2)$. In this problem there is a difference between the non-compatible case $n_0+|E_0|^2\neq 0$ and the compatible case $n_0+|E_0|^2=0$ concerning the rate of convergence as $\lambda\to\infty$, because the initial layer phenomenon occurs in the non-compatible case. But the difference between the both cases does not appear enough in (1). In the present paper we investigate precisely the convergence rate in λ of solutions for (Z_λ) and the effect of the initial layer so that we can understand the difference between the both cases more clearly. In fact, we have the following observation. If we perform a formal perturbation method under the assumptions

$$(2) \qquad E_\lambda=E^{(0)}+\lambda^{-1}E^{(1)}+\lambda^{-2}E^{(2)}+O(\lambda^{-3}),$$
$$(3) \qquad n_\lambda=n^{(0)}+\lambda^{-1}n^{(1)}+\lambda^{-2}n^{(2)}+O(\lambda^{-3}),$$

as $\lambda\to\infty$ in a suitable sense, with smooth functions $E^{(j)}$, $n^{(j)}$ on $\boldsymbol{R}\times\boldsymbol{R}^N$ independent of λ, we obtain

zeroth order equations
$$n^{(0)}=-|E^{(0)}|^2,\quad i\partial_t E^{(0)}+\Delta E^{(0)}=-|E^{(0)}|^2E^{(0)},$$
first order equations
$$n^{(1)}=-2\,\mathrm{Re}\,(\bar{E}^{(0)}\cdot E^{(1)}),$$
$$i\partial_t E^{(1)}+\Delta E^{(1)}=-|E^{(0)}|^2E^{(1)}-2\,(\mathrm{Re}(\bar{E}^{(0)}\cdot E^{(1)}))E^{(0)},$$
second order equations
$$\partial_t^2 n^{(0)}-\Delta n^{(2)}=\Delta(|E^{(1)}|^2+2\,\mathrm{Re}\,(\bar{E}^{(0)}\cdot E^{(2)})),$$
$$i\partial_t E^{(2)}+\Delta E^{(2)}=-|E^{(0)}|^2E^{(2)}-2(\mathrm{Re}\,(\bar{E}^{(0)}\cdot E^{(1)}))E^{(1)}+n^{(2)}E^{(0)},$$

with the initial data $n^{(0)}(0)=n_0$, $\partial_t n^{(0)}(0)=n_1$, $E^{(0)}(0)=E_0$, and $n^{(j)}(0)=0$, $E^{(j)}(0)=0$ for $j=1,2$. From the equation for $E^{(0)}$ we see that $E^{(0)}$ is equal to the solution E to (NLS). For the equation of $E^{(1)}$, we compute the time derivative of $\|E^{(1)}(t)\|_2^2$ and use Gronwall's inequality to obtain $E^{(1)}=0$, where $\|\cdot\|_2$ denotes the L^2-norm. This leads to

$$(4) \qquad E_\lambda=E+\lambda^{-2}E^{(2)}+O(\lambda^{-3}),$$
$$(5) \qquad n_\lambda+|E_\lambda|^2=\lambda^{-2}(n^{(2)}+2\,\mathrm{Re}\,(\bar{E}\cdot E^{(2)}))+O(\lambda^{-3}),$$

which is exactly the same condition as the one used in Gibbons [2] as the first step of his formal derivation of (NLS) from (Z_λ). Under the assumptions (2)–(3) it must hold that

$$n_0+|E_0|^2=0 \quad\text{and}\quad n_1+(\partial_t|E|^2)(0)=n_1+2\,\mathrm{Im}\,(\bar{E}_0\cdot\Delta E_0)=0.$$

This compatibility condition on the initial data is so strong that this is a main drawback of the formal perturbation method. But (4)–(5) suggest that $C_1\lambda^{-1}$ in the RHS of (1) can be replaced by $C_1\lambda^{-2}$ in that case. We now state our main result.

Theorem. *Let* $1\leq N\leq 3$. *Let* $E_0, n_0\in\mathcal{S}$ *and let* $n_1\in\mathcal{S}\cap\dot{H}^{-1}$. *Let* E

be the solution of (NLS) *with the maximal existence time* $T_{\max}$ *and let* (n_λ, E_λ) *be the solution of* (Z_λ). *Then*:

(1) *For any* T *with* $0<T<T_{\max}$ *and any* $m\in \boldsymbol{N}$ *there exist two positive constants* C *and* λ_0 *such that for any* $\lambda\geq\lambda_0$

$$\sup_{0\leq t\leq T}\|n_\lambda(t)+|E_\lambda(t)|^2-Q^{(1)}(\lambda t)-\lambda^{-1}Q^{(2)}(\lambda t)\|_{H^m}\leq C\lambda^{-1},$$

where $$Q^{(1)}(t)=(\cos t(-\Delta)^{1/2})(n_0+|E_0|^2)$$

and $$Q^{(2)}(t)=(-\Delta)^{-1/2}(\sin t(-\Delta)^{1/2})(n_1+2\,\mathrm{Im}\,(\bar{E}_0\cdot\Delta E_0)).$$

In particular, for any $\lambda\geq\lambda_0$

$$\sup_{0\leq t\leq T}\|n_\lambda(t)+|E_\lambda(t)|^2-Q^{(1)}(\lambda t)\|_{H^m}\leq C\lambda^{-1}.$$

(2) *Assume* $n_0+|E_0|^2\neq 0$. *Then for any* T *with* $0<T<T_{\max}$ *and any* $m\in\boldsymbol{N}$ *there exist two positive constants* C *and* λ_0 *such that for any* $\lambda\geq\lambda_0$

$$\sup_{0\leq t\leq T}\|E_\lambda(t)-E(t)\|_{H^m}\leq C\lambda^{-1}.$$

(3) *Assume* $n_0+|E_0|^2=0$. *In addition, when* $N\leq 2$, *assume that* n_1 *takes the form* $n_1=\nabla\cdot\phi$ *for some* $\phi\in\mathcal{S}$. *Then for any* T *with* $0<T<T_{\max}$ *and any* $m\in\boldsymbol{N}$ *there exist two positive constants* C *and* λ_0 *such that for any* $\lambda\geq\lambda_0$

$$\sup_{0\leq t\leq T}\|E_\lambda(t)-E(t)\|_{H^m}\leq C\lambda^{-2}.$$

Remark. (1) The assumption $n_1\in\mathcal{S}\cap\dot{H}^{-1}$ is redundant when $N=3$ since $\mathcal{S}\subset\dot{H}^{-1}$ for $N\geq 3$. This fact follows by using the Hardy inequality in the Fourier space.

(2) $Q^{(j)}(t)$, $j=1,2$, in part (1) solve the wave equation

$$\partial_t^2 Q^{(j)}-\Delta Q^{(j)}=0,\quad t>0,\ x\in\boldsymbol{R}^N,\ j=1,2$$

with the initial conditions $Q^{(1)}(0,x)=n_0(x)+|E_0(x)|^2$, $\partial_t Q^{(1)}(0,x)=0$, and $Q^{(2)}(0,x)=0$, $\partial_t Q^{(2)}(0,x)=n_1(x)+2\,\mathrm{Im}\,(\bar{E}_0\cdot\Delta E_0)(x)$, respectively. The terms $Q^{(1)}(\lambda t)$ and $\lambda^{-1}Q^{(2)}(\lambda t)$ represent the first initial layer and the second initial layer, respectively, in the nonlinear Schrödinger limit of (Z_λ).

(3) For $t>0$ $Q^{(1)}(\lambda t)$ tends to zero locally in space for $N=1$ and globally in space for $N=2,3$ as $\lambda\to\infty$. Accordingly, the above theorem implies that for $0<t<T_{\max}$, n_λ behaves like $-|E_\lambda|^2$ and so like $-|E|^2$ as $\lambda\to\infty$.

(4) The results in parts (2)–(3) are optimal concerning the rate of convergence with respect to λ. Indeed, there exist nontrivial solutions E_λ satisfying

$$\liminf_{\lambda\to\infty}\lambda\sup_{0\leq t\leq T}\|E_\lambda(t)-E(t)\|_{H^m}>0$$

in the non-compatible case and

$$\liminf_{\lambda\to\infty}\lambda^2\sup_{0\leq t\leq T}\|E_\lambda(t)-E(t)\|_{H^m}>0$$

in the compatible case.

The above theorem gives a detailed description of formation of initial layers with almost optimal rate of convergence for solutions of (Z_λ). For any T with $0<T<T_{\max}$, E_λ behaves like E on $[0,T]$ and n_λ behaves like $-|E|^2$ on $(0,T]$. This difference between the time intervals for convergence of these solutions is due to the initial layer phenomena. The formation of initial layers also reflects the rate of convergence for solutions of (Z_λ).

Our method of the proof of the theorem depends essentially on the special propagation properties of acoustic waves and of nonlinear Schrödinger waves. An outline of the proofs roughly given as follows. By setting $Q_\lambda = n_\lambda + |E_\lambda|^2$, (Z_λ) is rewritten as the system of integral equations

$$E_\lambda(t) = U(t)E_0 + i\int_0^t U(t-s)(|E_\lambda|^2 E_\lambda - Q_\lambda E_\lambda)(s)ds,$$

$$Q_\lambda(t) = Q^{(1)}(\lambda t) + \lambda^{-1}Q^{(2)}(\lambda t) + \lambda^{-1}\int_0^t (-\Delta)^{-1/2}(\sin \lambda(t-s)(-\Delta)^{1/2})\partial_s^2 |E_\lambda|^2(s)ds,$$

where $U(t) = e^{it\Delta}$ and $Q^{(j)}$ is as in the above theorem. Hence, we have

$$E_\lambda(t) - E(t) = i\int_0^t U(t-s)(|E_\lambda|^2 E_\lambda - |E|^2 E)(s)ds - i\int_0^t U(t-s)Q_\lambda(s)E_\lambda(s)ds. \tag{6}$$

Our main task is to obtain sharp estimates for the second integral in the RHS of (6). The term $Q_\lambda E_\lambda$ in the integrand corresponds to the interaction between acoustic wave Q_λ and nonlinear Schrödinger wave E_λ. Q_λ propagates according to the Huygens principle and is localized in a neighborhood of the sphere $|x| = \lambda t$. Therefore, Q_λ propagates very fact as $\lambda \to \infty$. On the other hand, E_λ propagates with group velocity independent of λ and is well localized in a ball with radius independent of λ. Thus, we can prove that the product $Q_\lambda E_\lambda$ converges to zero faster Q_λ as $\lambda \to \infty$. This is a main idea of our method, which is different from that of [1]. Detailed arguments will be given elsewhere.

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