28. Universal R-matrices for Quantum Groups Associated to Simple Lie Superalgebras

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Introduction. Let H be a Hopf algebra with coproduct $\Delta: H \to H \otimes H$. Let $\mathcal{R}=\sum_i a_i \otimes b_i \in H \otimes H$ be an invertible element. The triple $(H, \Delta, \mathcal{R})$ is called a *quasi-triangular Hopf algebra* if $\mathcal{R}$ satisfies the following properties (see [1]):

(0.1) $$\bar{\Delta}(x)=\mathcal{R}\Delta(x)\mathcal{R}^{-1} \qquad (x \in H),$$
$$(\Delta \otimes id)(\mathcal{R})=\mathcal{R}_{13}\mathcal{R}_{23}, \qquad (id \otimes \Delta)(\mathcal{R})=\mathcal{R}_{13}\mathcal{R}_{12}$$

where $\bar{\Delta}=\tau \circ \Delta$, $\tau(x \otimes y)=y \otimes x$ and $\mathcal{R}_{12}=\sum_i a_i \otimes b_i \otimes 1$, $\mathcal{R}_{13}=\sum_i a_i \otimes 1 \otimes b_i$, $\mathcal{R}_{23}=\sum_i 1 \otimes a_i \otimes b_i$. The $\mathcal{R}$ is called the *universal R-matrix*. From this definition, it follows that $\mathcal{R}$ satisfies the Yang-Baxter equation:

(0.2) $$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}=\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

Let $\mathcal{G}$ be a complex simple Lie algebra and $U(\mathcal{G})$ the universal enveloping algebra of $\mathcal{G}$. In 1985, Drinfeld [1] and Jimbo [2] associated to each $\mathcal{G}$, the h-adic topologically free $\boldsymbol{C}[[h]]$-Hopf algebra $(U_h(\mathcal{G}), \Delta)$ such that $U_h(\mathcal{G})/hU_h(\mathcal{G})=U(\mathcal{G})$, which is now called the *quantum group* or the *quantized enveloping algebra*. Moreover Drinfeld [1] gave a method of constructing an element $\mathcal{R}=U_h(\mathcal{G})\hat{\otimes}U_h(\mathcal{G})$ such that $(U_h(\mathcal{G}), \Delta, \mathcal{R})$ is a quasi-triangular Hopf algebra. His method is called the *quantum double construction*. By using this method, Rosso [9] gave an explicit formula of $\mathcal{R}$ for $\mathcal{G}=sl_n(\boldsymbol{C})$, and Kirillov-Reshetikhin [6], Levendorskii-Soibelman [8] gave such a formula for any $\mathcal{G}$.

Let $\tilde{\mathcal{G}}=\tilde{\mathcal{G}}_0 \oplus \tilde{\mathcal{G}}_1$ be a complex simple Lie superalgebra of types $A-G$ and $U(\tilde{\mathcal{G}})$ the universal enveloping superalgebra of $\tilde{\mathcal{G}}$. In this note, we associate to each $\tilde{\mathcal{G}}$, an h-adic topological $\boldsymbol{C}[[h]]$-Hopf superalgebra $(U_h(\tilde{\mathcal{G}}), \Delta^s)$ such that $U_h(\tilde{\mathcal{G}})/hU_h(\tilde{\mathcal{G}})=U(\tilde{\mathcal{G}})$. In fact, the definition of $U_h(\tilde{\mathcal{G}})$ depends on a choice of the Cartan matrix and the parities of the simple roots of $\tilde{\mathcal{G}}$. (For the terminologies *Lie superalgebra* and *Hopf superalgebra*, see [4, 6].) We also introduce an h-adic topological Hopf algebra $(U_h^\sigma(\tilde{\mathcal{G}}), \Delta^\sigma)$. The $U_h^\sigma(\tilde{\mathcal{G}})$ contains $U_h(\tilde{\mathcal{G}})$ as a subalgebra and the Hopf algebra structure of $(U_h^\sigma(\tilde{\mathcal{G}}), \Delta^\sigma)$ arises naturally from the Hopf superalgebra structure of $(U_h(\tilde{\mathcal{G}}), \Delta^s)$. In this note, by using the quantum double construction, we construct an element $\mathcal{R} \in U_h^\sigma(\tilde{\mathcal{G}}) \otimes U_h^\sigma(\tilde{\mathcal{G}})$ explicitly so that $(U_h^\sigma(\tilde{\mathcal{G}}), \Delta^\sigma, \mathcal{R})$ is a quasi-triangular Hopf algebra. In the process of constructing $\mathcal{R}$, we can also show that $U_h^\sigma(\tilde{\mathcal{G}})$ and $U_h(\tilde{\mathcal{G}})$ are topologically free.

Details omitted here will be published elsewhere.

After I finished this work, Professor E. Date informed me about the existence of the preprint [5] of Khoroskin and Tolstoy, whose results seem to have some overlap with the ones in the present paper.

§ 1. **Preliminaries.** Let $\tilde{\mathcal{G}}=\tilde{\mathcal{G}}_0\oplus\tilde{\mathcal{G}}_1$ be a simple Lie superalgebra of types $A-G$ and $U(\tilde{\mathcal{G}})=U(\tilde{\mathcal{G}})_0\oplus U(\tilde{\mathcal{G}})_1$ the universal enveloping superalgebra of $\tilde{\mathcal{G}}$. Let $\Pi=\{\alpha_1,\cdots\alpha_n\}$ be the set of simple roots and $p\colon \Pi\to\{0,1\}$ the parity function. Let A be a Cartan matrix related to Π. We assume that A is of distinguished type when $\tilde{\mathcal{G}}\neq sl(n|m)$ or $osp(n|m)$. In this note, we define, for such A an h-adic topologically free $\boldsymbol{C}[[h]]$-Hopf superalgebra $(U_h(\tilde{\mathcal{G}})=U_h(\tilde{\mathcal{G}},A,p)=U_h(\tilde{\mathcal{G}})_0\oplus U_h(\tilde{\mathcal{G}})_1,\Delta^s)$ such that $U_h(\tilde{\mathcal{G}})/hU_h(\tilde{\mathcal{G}})=U(\tilde{\mathcal{G}})$. (The Hopf superalgebra $U_h(\tilde{\mathcal{G}})$ corresponding to $osp(2|1)$ has already been introduced by Kulish and Reshetikhin [7].) Let $\boldsymbol{Z}/2\boldsymbol{Z}=\langle\sigma\rangle$ act on $U_h(\tilde{\mathcal{G}})$ by $\sigma\cdot x=(-1)^i x$ for $x\in U_h(\tilde{\mathcal{G}})_i$. Define an h-adic $\boldsymbol{C}[[h]]$-Hopf algebra as follows:

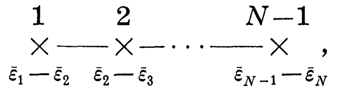
(i) $U_h^\sigma(\tilde{\mathcal{G}})=U_h(\tilde{\mathcal{G}})\hat{\otimes}\boldsymbol{C}[[h]]\langle\sigma\rangle$ as h-adic $\boldsymbol{C}[[h]]$-modules. We denote the element $x\otimes\sigma^c$ ($x\in U_h(\tilde{\mathcal{G}})$, $c\in\boldsymbol{Z}$) of $U_h^\sigma(\tilde{\mathcal{G}})$ by $x\sigma^c$.

(ii) The product of $U_h(\tilde{\mathcal{G}})$ is defined by $x\sigma^c\cdot x'\sigma^{c'}=(-1)^{ic}xx'\sigma^{c+c'}$ for $x\in U_h(\tilde{\mathcal{G}})$ and $x'\in U_h(\tilde{\mathcal{G}})_i$.

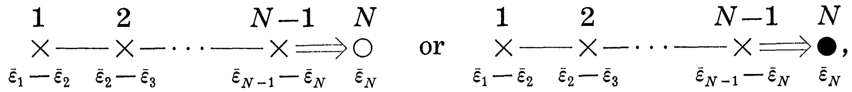
(iii) The coproduct Δ^σ is defined by putting $\Delta^\sigma(x\sigma^c)=\sum a_i\sigma^{p_i+c}\otimes b_i\sigma^c$ for $x\in U_h(\tilde{\mathcal{G}})$, $\Delta^s(x)=\sum a_i\otimes b_i$ and $b_i\in U_h(\tilde{\mathcal{G}})_{p_i}$. (The Hopf algebra $U_h^\sigma(\tilde{\mathcal{G}})$ corresponding to $sl(1|1)$ has already been introduced by Jing, Ge and Wu [3].)

§ 2. **Dynkin diagrams.** Let A be a Cartan matrix of rank n satisfying the condition of § 1. Let (Φ,Π) $(\Pi=(\alpha_1,\cdots,\alpha_n)$ be the root system of A where Φ and Π denote the set of roots and the set of simple roots. Put $N=n+1$ if A is of type A, and $N=n$ otherwise. We assume that (Φ,Π) is embedded in an N-dimensional complex linear space $\mathcal{S}$ with a non-degenerate symmetric bilinear form $(\ ,\ )$. Let $D=\mathrm{diag}\,(d_1,\cdots,d_n)$ be the diagonal matrix described below. They satisfy $DA=[(\alpha_i,\alpha_j)]$.

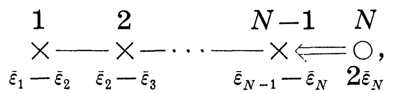
(i) *Types* A, B, C, D. Let $\{\bar{\varepsilon}_i\,|\,1\leq i\leq N\}$ be a basis of $\mathcal{S}$ such that $(\bar{\varepsilon}_i,\bar{\varepsilon}_j)=\pm 1$. We can arbitrarily choose the sign of $(\bar{\varepsilon}_i,\bar{\varepsilon}_i)$. In the diagram below, the element under the i-th vertex denotes the simple root α_i.

(A) 

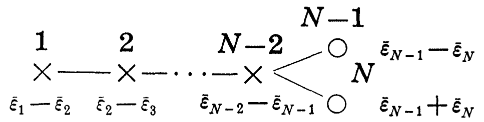
$D=\mathrm{diag}\,(1,\cdots,1)$.

(B) 

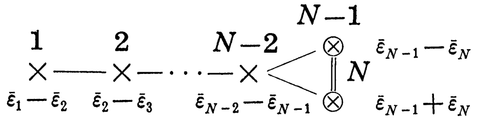
$D=\mathrm{diag}\,(1,\cdots,1,\tfrac{1}{2})$.

(C) 

$D=\mathrm{diag}\,(1,\cdots,1,2)$.

(D) 

if $(\bar{\varepsilon}_{N-1},\bar{\varepsilon}_{N-1})=(\bar{\varepsilon}_N,\bar{\varepsilon}_N)$,



if $(\bar\varepsilon_{N-1},\bar\varepsilon_{N-1})=-(\bar\varepsilon_N,\bar\varepsilon_N)$,

$D=\mathrm{diag}\,(1,\cdots,1)$.

(ii) *Types* F_4 *and* G_3. For type F_4 (resp. G_3), we normalize (,) by $(\alpha_3,\alpha_3)=2$ (resp. $(\alpha_3,\alpha_3)=-2$).

(F_4) 1 ○——4 ○⟹3 ○——2 ⊗, $D=\mathrm{diag}\,(2,1,1,2)$.

(G_3) 1 ⊗——3 ○⟸2 ○, $D=\mathrm{diag}\,(1,3,1)$.

We define the *parity function* $p: \Pi\to\{0,1\}$ as follows. If the i-th vertex is ○, ⊗ or ×, then we put $p(\alpha_i)=0$, 1 or $(2-|(\alpha_i,\alpha_i)|)/2$ respectively. If the N-th vertex of type B is ●, then we put $p(\bar\varepsilon_N)=1$.

Remark. In this note, we do not treat the Cartan matrix A of type $D(2,1;\alpha)$.

§3. **Definition of $U_h(\tilde{\mathcal{G}})$.** Let $((\Phi,\Pi),\mathcal{S})$ $p: \Pi\to\{0,1\}$ and $A=(a_{ij})_{1\le i,j\le n}$ be the root system, the parity function and the Cartan matrix given in §2. Let $\mathcal{H}=\mathcal{S}^*$ and identify the elements $H_\alpha\in\mathcal{H}$ and $\alpha\in\mathcal{S}$ by $\gamma(H_\alpha)=(\gamma,\alpha)$ $(\gamma\in\mathcal{S})$. Put $[X,Y]_v=XY-(-1)^{p(X)p(Y)}vYX$ and $[X,Y]=[X,Y]_1$ where $p(X)$ and $p(Y)$ are the parities of X and Y. Set

$$\begin{bmatrix} m+n \\ n\end{bmatrix}_t=\prod_{i=0}^{n-1}((t^{m+n-i}-t^{-m-n+i})/(t^{i+1}-t^{-i-1}))\in \boldsymbol{C}[t].$$

Put $q=e^{h/2}$ and $v_i=q^{(\bar\varepsilon_i,\bar\varepsilon_i)}$.

Let $U_h(\tilde{\mathcal{G}})=U_h(\tilde{\mathcal{G}},A,p)$ be the h-adic topological $\boldsymbol{C}[[h]]$-superalgebra generated by the $\boldsymbol{C}$-linear space $\mathcal{H}$ and the elements E_i, F_i $(1\le i\le n)$ with the parities $p(H)=0$ $(H\in\mathcal{H})$ and $p(E_i)=p(F_i)=p(\alpha_i)$ $(1\le i\le n)$ and defined by the following relations (3.1)–(3.5).

(3.1) $[H,H']=0$ for $H,H'\in\mathcal{H}$,

(3.2) $[H,E_i]=\alpha_i(H)E_i$, $[H,F_i]=-\alpha_i(H)F_i$ for $H\in\mathcal{H}$ and $1\le i\le n$,

(3.3) $[E_i,F_j]=\delta_{ij}sh(hH_{\alpha_i}/2)/sh(hd_i/2)$ for $1\le i,j\le n$,

(3.4) (i) $[E_i,E_j]=0$ for $1\le i,j\le n$ such that $a_{ij}=0$,

(ii) $\sum_{\nu=0}^{1+|a_{ij}|}(-1)^\nu\begin{bmatrix}1+|a_{ij}|\\ \nu\end{bmatrix}_{q^{d_i}}E_i^{1+|a_{ij}|-\nu}E_jE_i=0$ for $1\le i\ne j\le n$ and $p(\alpha_i)=0$,

(iii) $[[[E_i,E_j]_{v_j},E_k]_{v_{j+1}},E_j]=0$ for i ×——j ⊗——k × $(i<j<k)$, i ×——j ⊗⟹k ○ or i ×——j ⊗⟹k ●,

(iv) if A is of type B, $[[[E_{N-1},E_N]_{v_N},E_N],E_N]_{v_N^{-1}}=0$,

(v) if A is of type D, $[[E_{N-2},E_{N-1}]_{v_{N-1}},E_N]_{v_N}=[[E_{N-2},E_N]_{v_{N-1}},E_{N-1}]_{v_N}$,

(vi) if A is of type C, then $[E_{\varepsilon_{N-2}+\varepsilon_{N-1}},E_N]=0$, (resp. $[E_{2\varepsilon_{N-2}},E_{N-1}]=0$ or $[E_{\varepsilon_{N-3}+\varepsilon_{N-2}},E_{N-1}]=0$) for $N-2$ ×——$N-1$ ⊗⟸N ○

$N-2$ $N-1$ N $N-3$ $N-2$ $N-1$ N
(resp. ⊗——⊗⇐=○ or ×——○——⊗⇐=○)
where the elements $E_{\bar{\varepsilon}_{N-1}+\bar{\varepsilon}_N}$, $E_{2\bar{\varepsilon}_{N-2}}$ and $E_{\bar{\varepsilon}_{N-3}+\bar{\varepsilon}_{N-2}}$ will be defined in the following section.

(3.5) the relations (3.4) with E_i's replaced by F_i's.

A Hopf superalgebraic structure of $U_h(\tilde{\mathcal{G}})$ is given by a coproduct Δ^s defined by

$$\Delta^s(H)=H\otimes 1+1\otimes H \qquad \text{for } H\in\mathcal{H},$$
$$\Delta^s(E_i)=E_i\otimes 1+\exp\left(\frac{h}{2}H_{\alpha_i}\right)\otimes E_i \qquad \text{for } 1\leq i\leq n,$$
$$\Delta^s(F_i)=F_i\otimes\exp\left(-\frac{h}{2}H_{\alpha_i}\right)+1\otimes F_i \qquad \text{for } 1\leq i\leq n.$$

Lemma 3.1. *Put $C=\{H\in\mathcal{H}\mid\alpha_i(H)=0\ (1\leq i\leq n)\}$ and $U_h(\tilde{\mathcal{G}})'=U_h(\tilde{\mathcal{G}})/\overline{U_h(\tilde{\mathcal{G}})C}$. As a C-Hopf superalgebra, $U_h(\tilde{\mathcal{G}})'/hU_h(\tilde{\mathcal{G}})'$ is isomorphic to $U(\tilde{\mathcal{G}})$.*

By using the quantum double construction and Tanisaki's argument in [10] for $U_h^g(\tilde{\mathcal{G}})$, we can show the following theorem:

Theorem 3.2. *$U_h(\tilde{\mathcal{G}})$ is topologically free as an h-adic $\mathbf{C}[[h]]$-module.*

§4. Root vectors of $U_h(\tilde{\mathcal{G}})^+$ and $U_h(\tilde{\mathcal{G}})^-$. Let Φ_+ be the set of positive roots related to Π and let $\Phi'_+=\{\beta\in\Phi_+\mid\beta/2\notin\Phi\}$. For $\beta=c_1\alpha_1+\cdots+c_n\alpha_n\in\Phi'_+$, we put $l(\beta)=c_1+\cdots+c_n$, $c_i^\beta=c_i$ $(1\leq i\leq n)$, $g(\beta)=\min\{i\mid c_i\neq 0\}$ and $l'(\beta)=c_{g(\beta)}^{-1}l(\beta)\in\frac{1}{2}\mathbf{Z}$. We define a total order on Φ'_+ as follows. If $\alpha,\beta\in\Phi'_+$, we say that $\alpha\leq\beta$ if $g(\alpha)\leq g(\beta)$, $l'(\alpha)\leq l'(\beta)$, $c_{n-1}^\alpha\leq c_{n-1}^\beta$ and $c_n^\alpha\leq c_n^\beta$. Let $U_h(\tilde{\mathcal{G}})^+$ (resp. $U_h(\tilde{\mathcal{G}})^-$) be the unital subalgebra of $U_h(\tilde{\mathcal{G}})$ generated by E_i's (resp. F_i's).

Definition 4.1. For $\beta\in\Phi'_+$, we define the elements $E_\beta\in U_h(\tilde{\mathcal{G}})^+$ and $F_\beta\in U_h(\tilde{\mathcal{G}})^-$ as follows. For type F_4 (resp. G_3), E_{abcd} and E'_{abcd} (resp. E_{abc} and E'_{abc}) denote $E_{a\alpha_1+b\alpha_4+c\alpha_3+d\alpha_2}$ and $E'_{a\alpha_1+b\alpha_4+c\alpha_3+d\alpha_2}$ (resp. $E_{a\alpha_1+b\alpha_3+c\alpha_2}$ and $E'_{a\alpha_1+b\alpha_3+c\alpha_2}$).

(i) We put $E_{\alpha_i}=E_i$ $(1\leq i\leq n)$.

(ii) For $\alpha\in\Phi'_+$ such that $g(\alpha)<i$ and $\alpha+\alpha_i\in\Phi$, we put $E'_{\alpha+\alpha_i}=[E_\alpha,E_{\alpha_i}]_{q^{-(\alpha,\alpha_i)}}$. If A is of type B, $i=N$ and $\alpha=\bar{\varepsilon}_j$ $(1\leq j\leq N-1)$, let $E_{\alpha+\alpha_N}=(q^{1/2}+q^{-1/2})^{-1}E'_{\alpha+\alpha_N}$. If A is of type D, $i=N$ and $\alpha=\alpha_{N-1}$, let $E_{\alpha+\alpha_N}=(q+q^{-1})^{-1}E'_{\alpha+\alpha_N}$. If A is of type F_4, let $E_{1120}=(q+q^{-1})^{-1}E'_{1120}$ and $E_{1232}=(q^2+1+q^{-2})^{-1}E'_{1232}$. If A is of type G_3, let $E_{121}=(q+q^{-1})^{-1}E'_{121}$ and $E_{031}=(q^2+1+q^{-2})^{-1}E'_{031}$. Otherwise, put $E_\alpha=E'_\alpha$.

(iii) For $\alpha,\beta\in\Phi'_+$ such that $g(\alpha)=g(\beta)$, $\alpha<\beta$, $l'(\beta)-l'(\alpha)\leq 1$ and $\alpha+\beta\in\Phi'_+$, we put $E'_{\alpha+\beta}=[E_\alpha,E_\beta]_{q^{-(\alpha,\beta)}}$. If A is of type C (resp. D, F_4 of G_3), then $E_{\alpha+\beta}$ is defined by $(q+q^{-1})^{-1}E'_{\alpha+\beta}$ (resp. $(q+q^{-1})^{-1}E'_{\alpha+\beta}$, $(q^2+q^{-2})^{-1}E'_{\alpha+\beta}$ or $(q^2+1+q^{-2})^{-1}E'_{\alpha+\beta}$).

(iv) $F_\alpha\in U_h(\tilde{\mathcal{G}})^-$ $(\alpha\in\Phi'_+)$ is also defined in the same manner.

§5. The main result. For $\alpha\in\Phi'_+$, we define an integer d_α as follows. If $(\alpha,\alpha)=0$, we put $d_\alpha=1$. If A is of type G_3 and $\alpha=\alpha_1+2\alpha_3+\alpha_2$, we put $d_\alpha=2$. Otherwise, put $d_\alpha=|(\alpha,\alpha)|/2$.

Definition 5.1. Assume that A is of type A, B, C or D. For $\alpha=c_1\alpha_1+\cdots+c_n\alpha_n \in \Phi'_+$, let $p_\alpha=\sum_{p(i)=1}^{n} c_i$ where $p(i)$ denotes $p(\alpha_i)$. For $\alpha=\bar{\varepsilon}_i+n_j\bar{\varepsilon}_j \in \Phi'_+$ $(1\leq i\leq j\leq N,\ -1\leq n_j\leq 1)$, put $V_\alpha=(\prod_{i<l\leq N}(\bar{\varepsilon}_l,\bar{\varepsilon}_l)v_l)(\prod_{j<u\leq N}(\bar{\varepsilon}_u,\bar{\varepsilon}_u)v_u)^{n_j}$. For $\alpha\in\Phi'_+$, we define the element $X_\alpha\in C[[h]]^\times$ as follows. For A of type B (resp. C or D) and $\alpha=\bar{\varepsilon}_i+\bar{\varepsilon}_j$ $(1\leq i<j\leq N)$, we put $X_\alpha=(-1)^{p(N)}(\bar{\varepsilon}_N,\bar{\varepsilon}_N)$ (resp. v_N or $(\bar{\varepsilon}_{N+1},\bar{\varepsilon}_{N+1})v_{N+1}^{-1}$). For A of type D and $\alpha=2\bar{\varepsilon}_i$ $(1\leq i\leq N,\ p(\bar{\varepsilon}_i-\bar{\varepsilon}_N)=1)$, we put $X_\alpha=(\bar{\varepsilon}_{N+1},\bar{\varepsilon}_{N+1})$. Otherwise, put $X_\alpha=1$. We let $M_\alpha=(-1)^{\{p_\alpha(p_\alpha-1)/2\}}\cdot X_\alpha\cdot V_\alpha$.

Remark. In Definition 5.1, we assumed, for simplicity, A is of type A, B, C or D. M_α can also be defined in the F_4- and G_3-cases so that Theorem 5.2 below is also true in those cases. In all cases, $M_\alpha=(-1)^a q^b$ for some integers a and b.

Let $U_h^\sigma(\tilde{\mathcal{G}})$ be the h-adic $C[[h]]$-Hopf algebra defined in §1. For this $U_h^\sigma(\tilde{\mathcal{G}})$, we construct the universal R-matrix by using the quantum double construction. Let $\Phi_n(t)=\prod_{i=1}^{n}((1-t^i)/(1-t))$ and let $e(u;t)=\sum_n(u^n/\Phi_n(t))$ be the formal power series called the q-exponential. We note that $E_\alpha^2=0$ if $(\alpha,\alpha)=0$.

Theorem 5.2. *Let $\mathcal{R}$ be an element of $U_h^\sigma(\tilde{\mathcal{G}})\hat{\otimes}U_h^\sigma(\tilde{\mathcal{G}})$ defined by*

$$\mathcal{R}=\{\prod_{\alpha\in\Phi'_+} e((q^{d_\alpha}-q^{-d_\alpha})M_\alpha^{-1}E_\alpha\otimes F_\alpha\sigma^{p(\alpha)};\ (-1)^{p(\alpha)}q^{(\alpha,\alpha)})\}$$
$$\times\left\{\frac{1}{2}\sum_{c,c'\in\{0,1\}}(-1)^{cc'}\sigma^c\otimes\sigma^{c'}\right\}\cdot\exp\left(\frac{h}{2}t_0\right)$$

where $t_0=\sum_{i=1}^{N}H_i\otimes H_i$ and H_i's are basis elements of $\mathcal{H}$ such that $(H_i,H_j)=\delta_{ij}$. (The product over α is taken with respect to the total order $<$ defined in §4.) Then $(U_h^\sigma(\tilde{\mathcal{G}}),\Delta^\sigma,\mathcal{R})$ is a quasi-triangular Hopf algebra.

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