27. *Fermat Motives and the Artin-Tate Formula. I*

By Noriyuki SUWA
Department of Mathematics, Tokyo Denki University

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In this note, we mention some results on the Artin-Tate formula for Fermat motives in the higher dimensional cases, which was achieved by [4] and [10] in the 2-dimensional case. Detailed account will be published elsewhere.

1. Definition of Fermat motives (Shioda [4]). **1.1.** Let k be a field and let X be the Fermat variety of dimension n and of degree m over k:

$$X:\ T_0^m+T_1^m+\cdots+T_{n+1}^m=0\subset \boldsymbol{P}_k^{n+1}.$$

We assume that $(m,p)=1$ if k is of characteristic $p>0$. Let μ_m denote the group of m-th root of unity in $\bar{k}$. The group $G=(\mu_m)^{n+2}/(\text{diagonal})$ acts naturally on $X_{\bar{k}}=X\otimes_k\bar{k}$. The character group $\hat{G}$ of G is identified with the set

$$\left\{\boldsymbol{a}=(a_0,a_1,\cdots,a_{n+1})\,;\ a_i\in \boldsymbol{Z}/m,\ \sum_{i=0}^{n+1}a_i=0\right\};$$

Let $(\boldsymbol{Z}/m)^\times$ act on $\hat{G}$ by $t\boldsymbol{a}=(ta_0,\cdots,ta_{n+1})\in\hat{G}$ for any $\boldsymbol{a}\in\hat{G}$ and $t\in(\boldsymbol{Z}/m)^\times$.

Let ζ_m be a fixed primitive m-th root of unity in $\bar{\boldsymbol{Q}}$. For the $(\boldsymbol{Z}/m)^\times$-orbit A of $\boldsymbol{a}=(a_0,\cdots,a_{n+1})\in\hat{G}$, define

$$p_A=\frac{1}{m^{n+1}}\sum_{g\in G}\mathrm{Tr}_{\boldsymbol{Q}(\zeta_m^d)/\boldsymbol{Q}}(\boldsymbol{a}(g)^{-1})g\in \boldsymbol{Z}\left[\frac{1}{m}\right][G].$$

Here $d=\gcd(m,a_0,\cdots,a_{n+1})$. Then p_A are idempotents, i.e.

$$p_A\cdot p_B=\begin{cases}p_A & \text{if } A=B\\ 0 & \text{if } A\neq B\end{cases},\quad \sum_{A\in O(\hat{G})}p_A=1$$

where $O(\hat{G})$ denotes the set of $(\boldsymbol{Z}/m)^\times$-orbits in $\hat{G}$. The pair $M_A=(X,p_A)$ defines a motive over k, called the *Fermat submotive* of X corresponding to A (Shioda [4], p. 125).

1.2. Define a subset $\mathfrak{A}$ of $\hat{G}$ by

$$\mathfrak{A}=\{\boldsymbol{a}=(a_0,\cdots,a_{n+1})\in\hat{G}\,;\ a_i\neq 0 \text{ for all } i\}.$$

For each $\boldsymbol{a}\in\mathfrak{A}$, let

$$\|\boldsymbol{a}\|=\sum_{i=1}^{n+1}\left\langle\frac{a_i}{m}\right\rangle-1$$

where $\langle x\rangle$ stands for the fractional part of $x\in\boldsymbol{Q}/\boldsymbol{Z}$.

1.3. Let R be a ring, in which m is invertible, and let F be a contravariant functor from a category of varieties over k to the category of R-modules. For a Fermat submotive $M_A=(X,p_A)$ of X, define

$$F(M_A)=\mathrm{Im}\,[p_A^*: F(X)\to F(X)].$$

Example 1.4. Let l be prime number different from the characteristic of k. The l-adic étale cohomology groups $H^{\cdot}(X,\boldsymbol{Q}_l(i))$, $i\in\boldsymbol{Z}$; moreover, if l

is prime to m, $H^{\cdot}(X, Z/l^r(i))$, $H^{\cdot}(X, Z_l(i))$, $H^{\cdot}(X, Q_l/Z_l(i))$, $i \in Z$.

Example 1.5. The de Rham cohomology groups $H^{\cdot}_{DR}(X/k)$, or the Hodge spectral sequence $E_1^{i,j}=H^j(X, \Omega_X^i)\Rightarrow H_{DR}^{i+j}(X/k)$.

For the examples below, assume that k is perfect of characteristic $p>0$.

Example 1.6. The crystalline cohomology groups $H^{\cdot}(X/W_n)$, $H^{\cdot}(X/W)$, $H^{\cdot}(X/W)_K$, or the slope spectral sequences $E_1^{i,j}=H^j(X, W_n\Omega_X^i)\Rightarrow H^{\cdot}(X/W_n)$, and $E_1^{i,j}=H^j(X, W\Omega_X^i)\Rightarrow H^{\cdot}(X/W)$ (cf. [1], Ch. II).

Example 1.7. The logarithmic Hodge-Witt cohomology groups

$H^{\cdot}(X, Z/p^r(i))=H^{\cdot-i}(X, W_r\Omega^i_{X,\log})$, $H^{\cdot}(X, Z_p(i))=\varprojlim_r H^{\cdot}(X, Z/p^r(i))$ and $H^{\cdot}(X, Q_p/Z_p(i))=\varinjlim_r H^{\cdot}(X, Z/p^r(i))$, $i \in N$ (cf. [2], Ch. IV. 3, [9], Ch. I).

2. Fermat motives in characteristic $p>0$ ([10]). Throughout the section, X denotes the Fermat variety of dimension n and of degree m over $k=F_p$.

2.1. Let M_A be a Fermat submotive of X. We call the slopes and the Newton polygon of the F-crystal $(H^n(M_A/W), F)$ (cf. [3]) the *slopes* and the *Newton polygon* of M_A, respectively.

Definition 2.2. Let M_A be the Fermat submotive of X, corresponding to a $(Z/m)^{\times}$-orbit $A\subset\mathfrak{A}$.

(a) M_A is said to be *ordinary* if the Newton polygon and the Hodge polygon of M_A coincide.

(b) M_A is said to be *supersingular* if the Newton polygon has the pure slope $n/2$.

(c) M_A is said to be *of Hodge-Witt type* if $H^j(M_A, W\Omega^i)$ is of finite type over W for all pairs (i, j) with $i+j=n$ (cf. [2], Ch. IV, 4.6).

Proposition 2.3 ([10], Ch. II, 3). *Let M_A be the Fermat submotive of X, corresponding to a $(Z/m)^{\times}$-orbit $A\subset\mathfrak{A}$, and let f be the order of p in $(Z/m)^{\times}$.*

(*a*) *M_A is ordinary $\Leftrightarrow \|p\boldsymbol{a}\|=\|\boldsymbol{a}\|$ for each $\boldsymbol{a}\in A$ with $\|\boldsymbol{a}\|=i$, $0\le i\le (n-1)/2$.*

(b) *M_A is of Hodge-Witt type $\Leftrightarrow \|p^j\boldsymbol{a}\|-\|\boldsymbol{a}\|=0$, $\|p^j\boldsymbol{a}\|-\|\boldsymbol{a}\|=0, 1$ or $\|p^j\boldsymbol{a}\|-\|\boldsymbol{a}\|=0, -1$ for each $\boldsymbol{a}\in A$ with $\|\boldsymbol{a}\|=i$, $0\le i\le n/2-1$ and for each j, $0<j<f$.*

(c) *M_A is supersingular $\Leftrightarrow \sum_{j=0}^{f-1}\|p^j\boldsymbol{a}\|=nf/2$ for each $\boldsymbol{a}\in A$ with $\|\boldsymbol{a}\|=i$, $0\le i\le (n-1)/2$.*

Corollary 2.4. *The following conditions are all equivalent.*

(i) *M_A is ordinary and supersingular.*

(ii) *M_A is of Hodge-Witt type and supersingular.*

(iii) *$\|\boldsymbol{a}\|=n/2$ for each $\boldsymbol{a}\in A$.*

Remark 2.5. If X is defined over C, (iii) $\Leftrightarrow H^n(M_A, C)$ is purely of type $(n/2, n/2)$.

3. Supersingular Fermat motives. Throughout the section, $k=F_q$ of characteristic $p>0$, $\Gamma=\mathrm{Gal}(\bar{k}/k)$ and X denotes the Fermat variety of dimension $n=2r$ and of degree m over k.

3.1. Let Φ denote the geometric Frobenius of X relative to $k=\boldsymbol{F}_q$. Put

$$P_A(T)=\det(1-\Phi^*T\,|\,H^n(M_{A,\bar{k}},\boldsymbol{Q}_l))=\det(1-\Phi^*T\,|\,H^n(M_A/W)_K).$$

Then the decomposition $X=\oplus_A M_A$ defines a factorization

$$\det(1-\Phi^*T\,|\,H^n(X_{\bar{k}},\boldsymbol{Q}_l))=\det(1-\Phi^*T\,|\,H^n(X/W)_K)=\prod_A P_A(T).$$

Assume now that $k=\boldsymbol{F}_q$ contains all the m-th roots of unity. Then we have

$$P_A(T)=\prod_{\boldsymbol{a}\in A}(1-j(\boldsymbol{a})T)$$

for each $(\boldsymbol{Z}/m)^\times$-orbit $A\subset\mathfrak{A}$ (cf. Weil [8]). Here $j(\boldsymbol{a})$ denotes the Jacobi sum defined by

$$j(\boldsymbol{a})=(-1)^n\sum\chi(v_1)^{a_1}\cdots\chi(v_{n+1})^{a_{n+1}},$$

where the summation is taken over all the $(n+1)$-tuples $(v_1,\cdots,v_{n+1})\in(k^\times)^{n+1}$ subject to the relation $v_1+\cdots+v_{n+1}=-1$: $\boldsymbol{a}=(a_0,a_1,\cdots,a_{n+1})$ and $\chi: k^\times\to\bar{\boldsymbol{Q}}^\times$ is a multiplicative character of order m.

3.2. Let $CH^r(X)$ and $CH^r(X_{\bar{k}})$ denote the Chow group of rational equivalence classes of algebraic cycles of codimension r on X and $X_{\bar{k}}$, respectively. Recall that there is defined a cycle map $CH^r(X_{\bar{k}})\to H^n(X_{\bar{k}},\boldsymbol{Z}_l(r))$ for each prime l. The Tate conjecture ([6]) asserts that $H^n(X_{\bar{k}},\boldsymbol{Q}_l(r))^\Gamma$ is spanned by the image of the composite $CH^r(X)\to CH^r(X_{\bar{k}})\to H^n(X_{\bar{k}},\boldsymbol{Q}_l(r))$.

Note that it follows from Tate's theorem [7] together with the inductive structure of Fermat varieties [5] that the action of Φ^* on $H^n(X_{\bar{k}},\boldsymbol{Q}_l)$ is semi-simple.

Let $N^r(X_{\bar{k}})$ denote the group of numerical equivalence classes of algebraic cycles on $X_{\bar{k}}$ of codimension r. Then $N^r(X_{\bar{k}})$ is a free $\boldsymbol{Z}$-module of finite rank and equipped with a non-degenerate symmetric bilinear form induced by the intersection pairing. The decomposition $X=\oplus_A M_A$ defines decompostion:

$$CH^r(X)\otimes_{\boldsymbol{Z}}\boldsymbol{Z}\left[\frac{1}{m}\right]=\bigoplus_A CH^r(M_A)\otimes_{\boldsymbol{Z}}\boldsymbol{Z}\left[\frac{1}{m}\right],$$

$$CH^r(X_{\bar{k}})\otimes_{\boldsymbol{Z}}\boldsymbol{Z}\left[\frac{1}{m}\right]=\bigoplus_A CH^r(M_{A,\bar{k}})\otimes_{\boldsymbol{Z}}\boldsymbol{Z}\left[\frac{1}{m}\right]\quad\text{and}$$

$$N^r(X_{\bar{k}})\otimes_{\boldsymbol{Z}}\boldsymbol{Z}\left[\frac{1}{m}\right]=\bigoplus_A N^r(M_{A,\bar{k}})\otimes_{\boldsymbol{Z}}\boldsymbol{Z}\left[\frac{1}{m}\right].$$

Theorem 3.3. *Let X be the Fermat variety of dimension $n=2r$ and of degree m over k, M_A the Fermat submotive of X, corresponding to a $(\boldsymbol{Z}/m)^\times$-orbit $A\subset\mathfrak{A}$ and $P_A(T)=\prod_\alpha(1-\alpha T)$. Then we have implications*

$$[\text{(i)}\Leftrightarrow\text{(v)}\Leftrightarrow\text{(vi)}]\Leftarrow[\text{(ii)}\Leftrightarrow\text{(iii)}\Leftrightarrow\text{(iv)}]$$

among the following assertions. If the Tate conjecture holds true for X, these are all equivalent.

(i) *M_A is supersingular.*

(ii) *There is a prime $l\neq p$ such that the cycle map $CH^r(M_{A,\bar{k}})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_l\to H^n(M_{A,\bar{k}},\boldsymbol{Q}_l(r))$ is surjective.*

(iii) *For all primes* $l \neq p$, *the cycle map* $CH^r(M_{A,\bar{k}}) \otimes_{\mathbf{Z}} \mathbf{Q}_l \rightarrow H^n(M_{A,\bar{k}}, \mathbf{Q}_l(r))$ *is surjective.*

(iv) $N^r(M_{A,\bar{k}}) \otimes_{\mathbf{Z}} \mathbf{Z}[1/m] \neq 0$.

(v) α / q^r *is a root of unity for any* α.

(vi) α / q^r *is a root of unity for some* α.

Corollary 3.4. *If* M_A *is not supersingular, the cycle map* $CH^r(M_{A,\bar{k}}) \otimes_{\mathbf{Z}} \mathbf{Q}_l \rightarrow H^n(M_{A,\bar{k}}, \mathbf{Q}_l(r))$ *is zero.*

Corollary 3.5. *Assume that* m *is prime. Then* $B_n(X) - rk N^r(X_{\bar{k}})$ *is divisivle by* $m-1$.

Corollary 3.6. *We have*

$$rk N^r(X_{\bar{k}}) \leq 1 + \sharp A,$$

where the summation is taken over all the $(\mathbf{Z}/m)^{\times}$*-orbit* $A \subset \mathfrak{A}$ *such that* M_A *is supersingular. If the Tate conjecture holds true for* X *we have the equality*

$$rk N^r(X_{\bar{k}}) = 1 + \sharp A.$$

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