27. *On the Gaps between the Consecutive Zeros of the Riemann Zeta Function*

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§1. **Introduction.** Let γ_n be the n-th positive imaginary part of the zeros of the Riemann zeta function $\zeta(s)$. We have shown in [1] and [2] that for each integer $k\geq 1$ and for $T>T_0$,

$$C_1\frac{T\log T}{\log^k T}\leq \sum_{T\leq\gamma_n\leq 2T}(\gamma_{n+1}-\gamma_n)^k\leq C_2\frac{T\log T}{\log^k T},$$

where C_1 and C_2 are some positive constants. The implicit constant C_2 might be large. The purpose of the present article is to get an explicit C_2 (for the case $k=2$) under the assumption of the Riemann Hypothesis. We shall prove the following theorem.

Theorem 1. *For $T>T_0$, we have*

$$\sum_{\gamma_n\leq T}(\gamma_{n+1}-\gamma_n)^2\leq 9\cdot\frac{2\pi T}{\log\frac{T}{2\pi}}.$$

We shall prove this theorem as an application of the following mean value theorem which has been proved in [5]. We put

$$S(t)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+it\right)$$

and

$$F(a)=F(a,T)=\left(\frac{T}{2\pi}\log\frac{T}{2\pi}\right)^{-1}\sum_{0<\gamma,\gamma'\leq T}\left(\frac{T}{2\pi}\right)^{ia(\gamma-\gamma')}\frac{4}{4+(\gamma-\gamma')^2},$$

where γ and γ' run over the imaginary parts ($\neq 0$) of the zeros of $\zeta(s)$.

Theorem 2. *Suppose that $0<\Delta=o(1)$. Then we have for $T>T_0$,*

$$\int_0^T(S(t+\Delta)-S(t))^2dt$$
$$=\frac{T}{\pi^2}\left\{\int_0^{\Delta\log(T/2\pi)}\frac{1-\cos(a)}{a}da+\int_1^\infty\frac{F(a)}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\right\}+o(T).$$

In fact, we shall use it in the following form.

Corollary. *Suppose that $T>T_0$ and $1/\log(T/2\pi)\leq\Delta=o(1)$. Then we have with $|\theta|\leq 1$ and the Euler constant C_0,*

$$\int_0^T(S(t+\Delta)-S(t))^2dt=\frac{T}{\pi^2}\log\left(\Delta\log\frac{T}{2\pi}\right)$$
$$+\frac{T}{\pi^2}\left(C_0-\frac{\sin(\Delta\log(T/2\pi))}{\Delta\log\frac{T}{2\pi}}+2\theta\left(\frac{1}{\left(\Delta\log\frac{T}{2\pi}\right)^2}+2\right)+o(1)\right).$$

To get corollary from Theorem 2, we notice that

$$\int_0^{\Delta \log (T/2\pi)} \frac{1-\cos(a)}{a}\,da = \log\left(\Delta \log \frac{T}{2\pi}\right) + C_0 - Ci\left(\Delta \log \frac{T}{2\pi}\right)$$

and that by Theorem 2 of Goldston [6],

$$\left|\int_1^\infty \frac{F(a)}{a^2}\left(1-\cos\left(a\Delta \log \frac{T}{2\pi}\right)\right)da\right| \le 2\int_1^\infty \frac{F(a)}{a^2}\,da < 4.$$

To prove Theorem 2, we have applied Goldston's work [6] and also used the following lemma.

Lemma. *Suppose that $a \ll T^A$ with some positive constant A. Then we have*

$$W \equiv \sum_{0<\gamma\le T,\, \gamma+a>0} S(\gamma+a) \ll T \log T.$$

We shall give its proof below, using Selberg's explicit formula for $S(t)$ and the author's recent results [3] on the distribution of the zeros of $\zeta(s)$.

§ 2. Proof of Theorem 1. Suppose that

$$\frac{3}{2\log \frac{T}{2\pi}} \le H = O\left(\frac{1}{\log\log T}\right).$$

Then using the first formula in the introduction, we get

$$\begin{aligned} M &\equiv \sum_{\gamma_n \le T} (\gamma_{n+1}-\gamma_n)^2 \le \sum_{\substack{T/\log^2 T \le \gamma_n \le T \\ \gamma_{n+1}-\gamma_n \ge H}} (\gamma_{n+1}-\gamma_n)^2 + H^2 \frac{T}{2\pi}\log\frac{T}{2\pi} + O\left(\frac{T}{\log^3 T}\right) \\ &= \int_H^{C/\log\log T} \sum_{\substack{T/\log^2 T \le \gamma_n \le T \\ \gamma_{n+1}-\gamma_n \ge y}} (\gamma_{n+1}-\gamma_n)\,dy + H \sum_{\substack{T/\log^2 T \le \gamma_n \le T \\ \gamma_{n+1}-\gamma_n \ge H}} (\gamma_{n+1}-\gamma_n) \\ &\quad + H^2 \frac{T}{2\pi}\log\frac{T}{2\pi} + O\left(\frac{T}{\log^3 T}\right) \\ &= M_1 + M_2 + H^2 \frac{T}{2\pi}\log\frac{T}{2\pi} + O\left(\frac{T}{\log^3 T}\right), \quad \text{say.} \end{aligned}$$

Now for $H \le y \le (C/\log\log T)$,

$$\begin{aligned} \sum_{\substack{T/\log^2 T \le \gamma_n \le T \\ \gamma_{n+1}-\gamma_n \ge y}} (\gamma_{n+1}-\gamma_n) &\le 3 \sum_{\substack{T/\log^2 T \le \gamma_n \le T \\ \gamma_{n+1}-\gamma_n \ge y}} \left(\gamma_{n+1}-\gamma_n - \frac{2}{3}y\right) \\ &\le 3 \sum_{\substack{T/\log^2 T \le \gamma_n \le T \\ \gamma_{n+1}-\gamma_n \ge y}} \frac{1}{\left(\frac{2}{3}y\frac{1}{2\pi}\log\frac{T}{2\pi}\right)^2} \int_{\gamma_n}^{\gamma_{n+1}-(2/3)y} \left(S\left(t+\frac{2}{3}y\right)-S(t)\right)^2 dt \\ &\quad + O\left(\frac{T(\log\log T)^2}{y^2\log^3 T}\right) \\ &\le \frac{3}{\left(\frac{2}{3}y\frac{1}{2\pi}\log\frac{T}{2\pi}\right)^2} \int_0^T \left(S\left(t+\frac{2}{3}y\right)-S(t)\right)^2 dt + O\left(\frac{T(\log\log T)^2}{y^2\log^3 T}\right). \end{aligned}$$

Using the above corollary, we get

$$M_1 + M_2 \le \frac{12HT}{\left(\frac{2}{3}H\log\frac{T}{2\pi}\right)^2}\left\{2\log\left(\frac{2}{3}H\log\frac{T}{2\pi}\right) + 2C_0 + 9 - \frac{\sin\left(\frac{2}{3}H\log\frac{T}{2\pi}\right)}{\frac{2}{3}H\log\frac{T}{2\pi}}\right.$$

$$+\frac{11/3}{\left(\frac{2}{3}H\log\frac{T}{2\pi}\right)^2}+\frac{6}{\left(\frac{2}{3}H\log\frac{T}{2\pi}\right)^3}+o(1)\Big\}+O\left(T\left(\frac{\log\log T}{\log T}\right)^2\right).$$

Putting $H=B/\log(T/2\pi)$ and taking $B=10$, we get

$$M\leq\frac{2\pi T}{\log\frac{T}{2\pi}}\frac{1}{4\pi^2}\left\{B^2+\frac{54\pi}{B}\left(2\log\frac{2B}{3}+2C_0+9-\frac{3}{2B}\sin\left(\frac{2}{3}B\right)+\frac{33}{4B^2}+\frac{81}{4B^3}\right)+o(1)\right\}$$

$$\leq 8.55\times\frac{2\pi T}{\log\frac{T}{2\pi}}.$$

We may remark here that we can estimate, in a similar manner, the sum $\sum_{\gamma_n\leq T}((\gamma_{n+r}-\gamma_n)/r)^2$ for each integer $r\geq 2$.

§3. **Proof of Lemma.** We use the following explicit formula for $S(t)$ due to Selberg [10] (cf. 14.21 of Titchmarsh [11]). Let $\Lambda(n)$ be the von Mangoldt function. Then

$$S(t)=-\frac{1}{\pi}\sum_{n<Y^2}\frac{\Lambda_Y(n)\sin(t\log n)}{n^{\sigma_1}\log n}+O\left(\frac{1}{\log Y}\left|\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma_1+it}}\right|\right)+O\left(\frac{\log t}{\log Y}\right),$$

where $t>2$, $4\leq Y\leq t^2$, $\sigma_1=(1/2)+(1/\log Y)$ and

$$\Lambda_Y(n)=\begin{cases}\Lambda(n) & \text{for } 1\leq n\leq Y\\ \Lambda(n)\dfrac{\log(Y^2/n)}{\log Y} & \text{for } Y\leq n\leq Y^2.\end{cases}$$

We use this with $Y=T^b$, $0<b<1/2$. We may suppose that $a\leq\sqrt{Y}$, since otherwise we may replace $\sqrt{Y}-a$ by $\max(\sqrt{Y}-a,0)$ in the argument below.

$$\begin{aligned}W&=\sum_{\sqrt{Y}-a<\gamma\leq T}S(\gamma+a)+O(T\log T)\\&=-\frac{1}{\pi}\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma_1}\log n}\sum_{\sqrt{Y}-a<\gamma\leq T}\sin((\gamma+a)\log n)\\&\quad+O\left(\frac{1}{\log Y}\sum_{\sqrt{Y}-a<\gamma\leq T}\left|\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma_1+i(\gamma+a)}}\right|\right)\\&\quad+O\left(\frac{1}{\log Y}\sum_{\sqrt{Y}-a<\gamma\leq T}\log(\gamma+a)\right)+O(T\log T)\\&=W_1+W_2+W_3+O(T\log T),\quad\text{say.}\end{aligned}$$

Using theorem of [3], we get

$$\begin{aligned}W_1&\ll\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma_1}\log n}\left|\sum_{\sqrt{Y}-a<\gamma\leq T}n^{i\gamma}\right|\\&\ll\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma_1}\log n}\left(T\frac{\Lambda(n)}{\sqrt{n}}+\sqrt{n}\log T\log\log T\right)\ll T\log T.\end{aligned}$$

$$\begin{aligned}W_2&\ll\frac{1}{\log Y}\sqrt{T\log T}\left\{\sum_{\sqrt{Y}-a<\gamma\leq T}\left|\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma_1+i(\gamma+a)}}\right|^2\right\}^{1/2}\\&\ll\frac{1}{\log Y}\sqrt{T\log T}\left\{T\log T\sum_{n<Y^2}\frac{\Lambda_Y^2(n)}{n^{2\sigma_1}}+\sum_{m<n<Y^2}\frac{\Lambda_Y(m)\Lambda_Y(n)}{(mn)^{\sigma_1}}\right.\\&\quad\left.\times\left|\sum_{\sqrt{Y}-a<\gamma\leq T}\left(\frac{n}{m}\right)^{i\gamma}\right|\right\}^{1/2}\end{aligned}$$

$$=\frac{1}{\log Y}\sqrt{T\log T}\{W_4+W_5\}^{1/2}, \quad \text{say}.$$

Using theorem of [3] again, we get

$$W_5 \ll T\sum_{m<n<Y^2}\frac{\Lambda(n/m)\Lambda_Y(m)\Lambda_Y(n)}{n}+\sum_{m<n<Y^2}\frac{\Lambda_Y(m)\Lambda_Y(n)}{\sqrt{mn}}\frac{\log T}{\log\frac{n}{m}}$$

$$+\sum_{m<n<Y^2}\frac{\Lambda_Y(m)\Lambda_Y(n)}{m}\log T+\sum_{m<n<Y^2}\frac{\Lambda_Y(m)\Lambda_Y(n)}{\sqrt{mn}}\left(\frac{n}{m}\right)^{1/\log\log T}\frac{\log^2 T}{\log\log T}$$

$$+\sum_{m<n<Y^2}\frac{\Lambda_Y(m)\Lambda_Y(n)}{n}\sum_{\frac{1}{2}(n/m)<k<2(n/m),\,k\neq(n/m)}\Lambda(k)\frac{1}{\left|\log\left(\frac{n}{mk}\right)\right|}.$$

The last sum is

$$\ll \log T\sum_{d<2Y^2}\frac{1}{d}\Big(\sum_{mk=d}\Lambda(k)\Lambda_Y(m)\Big)\sum_{\substack{\frac{1}{2}d<n<2d\\ d\neq n}}\frac{1}{\left|\log\left(\frac{n}{d}\right)\right|}\ll T.$$

Treating the other sums similarly, we get

$$W_5 \ll T\log^3 T.$$

Since $W_4 \ll T\log^3 T$, we get $W_2 \ll T\log T$.

Since $W_3 \ll T\log T$, we get

$$W \ll T\log T.$$

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