26. The Modulo 2 Homology Group of the Space of Rational Functions

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§1. **Introduction.** Let $\hat{M}_k$ be the moduli space of $SU(2)$ monopoles associated with Yang-Mills-Higgs and Bogomol'nyi equations. It is shown [1] that $\hat{M}_k$ is homeomorphic to the space of based holomorphic maps of degree k from S^2 to S^2.

More generally we define $F_k^*(S^2, CP^m)$ to be the space of based holomorphic maps of degree k from S^2 to CP^m.

Segal [3] studied the connection between $F_k^*(S^2, CP^m)$ and $\Omega_k^2 CP^m$. The result is as follows

Theorem 1 [3]. *The inclusion*

$$i: F_k^*(S^2, CP^m) \longrightarrow \Omega_k^2 CP^m$$

is a homotopy equivalence up to dimension $k(2m-1)$, *the induced homomorphism* $i_*: \pi_q(F_k^*(S^2, CP^m)) \to \pi_q(\Omega_k^2 CP^m)$ *is bijective for* $q<k(2m-1)$ *and surjective for* $q=k(2m-1)$.

It is well know [2] that $\coprod_k \Omega_k^2 CP^m$ has natural loop sum and C_2 structure.

Recently Boyer and Mann [1] introduced loop sum and C_2 structure in $\coprod_k F_k^*(S^2, CP^m)$ which are compatible with the inclusion i. Hence we can naturally define the loop sum and the Dyer-Lashof operation Q_1 in $\oplus_k H_*(F_k^*(S^2, CP^m); Z_2)$.

By using these methods, Boyer and Mann produced certain elements in $H_*(F_k^*(S^2, CP^m); Z_2)$ some of which have degree greater than $k(2m-1)$. (cf. Theorem 1.)

Then the following question arises naturally.

Question. Are the elements produced by the loop sum and the Dyer-Lashof operation from ι_{2m-1} (ι_{2m-1} will be defined later) the basis of $H_*(F_k^*(S^2, CP^m); Z_2)$?

We shall study this question. The results are as follows. We write F_k^* for $F_k^*(S^2, CP^1)$.

Theorem A. *The elements produced by the loop sum and the Dyer-Lashof operation from* ι_1 *are the basis of* $H_*(F_2^*; Z_2)$.

Theorem B. *For* $m \geqq 2$, *the elements produced by the loop sum and the Dyer-Lashof operation from* ι_{2m-1} *are the basis of* $H_*(F_2^*(S^2, CP^m); Z_2)$.

Theorem C. *For* $m \geqq 2$, *the elements produced by the loop sum and the Dyer-Lashof operation from* ι_{2m-1} *are the basis of* $H_*(F_3^*(S^2, CP^m); Z_2)$.

Theorem D. *For* $m \geqq k+1$, *the elements produced by the loop sum* *and the sum and the Dyer-Lashof operation from* ι_{2m-1} *are the basis of* $H_*(F_k^*(S^2, CP^m); Z_2)$.

If we regard a function belonging to F_k^* as a holomorphic function $f: S^2 \to S^2$ of degree k such that $f(\infty)=1$ then F_k^* can be described in the following form.

$$F_k^* = \left\{ \frac{p(z)}{q(z)} = \frac{z^k + a_1 z^{k-1} + \cdots + a_k}{z^k + b_1 z^{k-1} + \cdots + b_k}; \ p(z) \text{ and } q(z) \text{ have no common root.} \right\}$$

Similarly we can assume $F_k^*(S^2, CP^m)$ as follows.

$F_k^*(S^2, CP^m) = \{[p_0(z), p_1(z), \cdots, p_m(z)]$; $p_i(z)$ are monic polynomials of degree k such that there exists no $\alpha \in C$ which satisfies $p_0(\alpha)=0$, $p_1(\alpha)=0, \cdots, p_m(\alpha)=0.\}$

Then it is clear that $F_1^*(S^2, CP^m)$ is homotopically equivalent to S^{2m-1}.

Before proving our results, we review the results of [1]. As for $H_*(\Omega^2 CP^m; Z_2)$, the following is well known.

Proposition 2 [2].

$$H_*(\Omega^2 CP^m; Z_2) = Z_2[\tilde{\iota}_{2m-1}, Q_{I_1}(\tilde{\iota}_{2m-1})] \otimes Z_2[Z]$$

where $Q_{I_1}(\tilde{\iota}_{2m-1}) = Q_1 \cdots Q_1(\tilde{\iota}_{2m-1})$ *(*I_1 *has length* 1 *and* 1 *is an any element of* N*) and* $\tilde{\iota}_{2m-1}$ *is the* mod 2 *reduction of the generator of* $H_{2m-1}(\Omega_1^2 CP^m; Z) = \pi_{2m-1}(\Omega_1^2 CP^m) = Z$.

Let ι_{2m-1} be the generator of $H_{2m-1}(F_1^*(S^2, CP^m); Z_2) = Z_2$. By operating the loop sum and Q_{I_1} to ι_{2m-1}, we obtain elements in $H_*(F_k^*(S^2, CP^m); Z_2)$. Then by using Proposition 2, we can easily prove the following proposition.

Proposition 3. *Let* ξ *be an element of* $H_q(\Omega_k^2 CP^m; Z_2)$ *for* $q \leq k(2m-1)$, *then we can construct an element* ζ *of* $H_q(F_k^*(S^2, CP^m); Z_2)$ *by the loop sum and the Dyer-Lashof operation from* ι_{2m-1} *such that* $i_*\zeta = \xi$.

In § 2 we shall prove Theorem A and in § 3 we shall prove Theorem D in the case $k=3$. The proof of Theorems B, C and Theorem D in the case $k \geq 4$ are omitted.

§ 2. Proof of Theorem A. In the following, all cohomology group and compact support cohomology group are assumed to be modulo 2 coefficients.

We define $R: F_2^* \to C^\times$ as follows. Let $p(z)/q(z)$ be an element of F_2^* and let α_1, α_2 be the roots of $p(z)$, β_1, β_2 be the roots of $q(z)$. Then $R(p(z)/q(z))$ is defined by $\prod_{i,j}(\alpha_i - \beta_j)$. Let Y_2 be $R^{-1}(1)$. Then it is shown in [3] that $R: F_2^* \to C^\times$ is a fiber bundle with simply connected fiber Y_2.

First we shall compute $H^*(Y_2)$. We define the closed subspace Y_1 of Y_2 as follows.

$$Y_1 = \left\{ \frac{p(z)}{q(z)} \in Y_2; \ q(z) \text{ has a multiple root.} \right\}$$

Because of the exact sequence

$$\cdots \longrightarrow H_c^q(Y_2 - Y_1) \longrightarrow H_c^q(Y_2) \longrightarrow H_c^q(Y_1) \longrightarrow H_c^{q+1}(Y_2 - Y_1) \longrightarrow \cdots$$

it will be enough to compute $H_c^*(Y_2 - Y_1)$ and $H_c^*(Y_1)$. Here H_c^* denotes the compact support cohomology.

Assertion 1. Y_1 is homeomorphic to $C^2 \coprod C^2$.

Let $\tilde{C}_2$ be the space of ordered distinct 2-tuples in $\boldsymbol{C}$. We think of $\boldsymbol{C}^\times$ as $\{(\xi_1, \xi_2) \in (\boldsymbol{C}^\times)^2 ; \xi_1\xi_2 = 1\}$.

Assertion 2. $Y_2 - Y_1$ is the quotient of $\boldsymbol{C}^\times \times \tilde{C}_2$ by a free action of the symmetric group Σ_2.

Now by using the compact support cohomology exact sequence and the Poincaré duality, we see

$$H^q(Y_2) = \begin{cases} \boldsymbol{Z}_2 & q=0,2 \\ 0 & \text{otherwise.} \end{cases}$$

By using the Serre spectral sequence of the above fiber bundle, we can prove Theorem A.

§3. **Proof of Theorem D in the case $k=3$.** We write X_3 for $F_3^*(S^2, \boldsymbol{CP}^m)$. To prove Theorem D in the case $k=3$, it will be enough to determine $H_c^q(X_3)$ for $q \leqq 9$ by Theorem 1 and Proposition 3. We define the closed subspace X_2 of X_3 and the closed subspace X_1 of X_2 as follows.

$$X_2 = \{[p_0(z), \cdots, p_m(z)] ; p_0(z) \text{ has a multiple root.}\}$$
$$X_1 = \{[p_0(z), \cdots, p_m(z)] ; p_0(z) \text{ has a triple root.}\}$$

Assertion 1. X_1 is homotopically equivalent to S^{2m-1}.

Assertion 2. $X_2 - X_1$ is homotopically equivalent to $(S^{2m-1})^2 \times S^1$.

By using the compact support cohomology exact sequence of the pair of spaces (X_2, X_1), we see $H_c^q(X_2)=0$ for $q \leqq 9$.

Let $\tilde{C}_3$ be the space of ordered distinct 3-tuples in $\boldsymbol{C}$.

Assertion 3. $X_3 - X_2$ is homotopically equivalent to $(S^{2m-1})^3 \times_{\Sigma_3} \tilde{C}_3$.

Assertion 4. $H^q((S^{2m-1})^3 \times_{\Sigma_3} \tilde{C}_3) = \begin{cases} \boldsymbol{Z}_2 & q=6m-3,\ 6m-2 \\ 0 & q \geqq 6m-1. \end{cases}$

Assertion 4 is proved by using the Serre spectral sequence of the following fiber bundle and the fact [2] $H^*(\tilde{C}_3/\Sigma_3) = H^*(S^1)$.

$$(S^{2m-1})^3 \longrightarrow (S^{2m-1})^3 \times_{\Sigma_3} \tilde{C}_3 \longrightarrow \tilde{C}_3/\Sigma_3.$$

Theorem D in the case $k=3$ can be deduced from the compact support cohomology exact sequence of the pair of spaces (X_3, X_2).

References

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