24. Convergence of a Class of Discrete Cubic Interpolatory Splines

By S. S. RANA
Freie Universität Berlin, Institut für Mathematik, West Germany

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1. Introduction. Discrete cubic splines which interpolate given functional values at one point lying in each mesh interval of a uniform mesh have been studied in [2]. The case in which these points of interpolation coincide with the mesh points of a nonuniform mesh was studied earlier by Lyche [4], [5]. For further results in this direction reference may be made to Dikshit and Rana [3]. In order to obtain the sharp convergence properties, we study in the present paper the problem of one point interpolation by discrete splines when the interpolatory points are not necessarily equispaced. The results, obtained in this paper include in particular some earlier results due to Lyche [5] for uniform mesh, Dikshit and Powar [2] and Chatterjee and Dikshit [1].

2. Existence and uniqueness. Let $P: a=x_0<x_1<\cdots<x_n=b$ denote a partition of $[a, b]$ with equidistant mesh points so that $p=x_i-x_{i-1}$ for all i. For a given $h>0$, suppose a real function $s(x, h)$ defined over $[a, b]$ and its restriction on $[x_{i-1}, x_i]$ is a polynomial s_i of degree 3 or less for $i=1, 2, \cdots, n$. Then $s(x, h)$ defines a discrete cubic spline if

(2.1) $$(s_{i+1}-s_i)(x_i+jh)=0,\ j=-1, 0, 1;\ i=1, 2, \cdots, n-1.$$

For an equivalent definition of a discrete cubic spline we introduce the difference operator

$$D_h^{\{0\}}f(x)=f(x);\ D_h^{\{1\}}f(x)=(f(x+h)-f(x-h))/2h;$$
$$D_h^{\{2\}}f(x)=(f(x+h)-2f(x)+f(x-h))/h^2.$$

We also use basic polynomials $x^{\{j\}}$ given by

$$x^{\{j\}}=x^j,\ j=0, 1, 2;\ x^{\{3\}}=x(x^2-h^2)$$

and observe that the condition (2.1) has the following equivalent form

(2.2) $$D_h^{\{j\}}s_i(x_i, h)=D_h^{\{j\}}s_{i+1}(x_i, h),\ j=0, 1, 2;\ i=1, 2, \cdots, n-1.$$

The class of all discrete cubic splines on P is denoted by $D(3, P, h)$ whereas $D_1(3, P, h)$ denotes the class of all $b-a$ periodic discrete cubic splines of $D(3, P, h)$.

We suppose that $\langle\theta_i\rangle_{i=1}^{\infty}$ is a real periodic sequence with period n so that $\theta_i=\theta_{i+n}$, $i=1, 2, \cdots$. Considering the points $y_i=x_{i-1}+\theta_i p$, $0\le\theta_i\le 1$, $i=1, 2, \cdots, n$, we propose the following.

Problem 1. Given $h>0$, for what restrictions on $\langle\theta_i\rangle$ does there exist a unique spline $s(x, h)\in D_1(3, P, h)$ satisfying the interpolatory condition

(2.3) $$s(y_i, h)=f(y_i),\ i=1, 2, \cdots, n,$$

where $\{f(y_i)\}$ is a given sequence of functional values?

For any function g of θ_i, θ_{i+1} and θ_{i+2} for all i, we write for convenience $\bar{g}$ for the function obtained from g by interchanging θ_{i+1} with $\theta^*_{i+1}=1-\theta_{i+1}$ and θ_i with θ^*_{i+2}.

Since $s(x,h)\in D_1(3,P,h)$, therefore for the interval $[x_{i-1},x_i]$, we may write

(2.4) $$6ps(x,h)=(x_i-x)^{\{3\}}M_{i-1}+(x-x_{i-1})^{\{3\}}M_i+6p(x-y_i)c_i+6pd_i$$

where $M_i=M_i(h)=D_h^{\{2\}}s(x_i,h)$ and c_i, d_i are appropriate constants. We are now set to answer the Problem 1 in the following:

Theorem 1. *Suppose $0<h\le p$. Then there exists a unique periodic spline $s(x,h)$ in the class $D_1(3,P,h)$ satisfying the interpolatory condition* (2.3), *if either* (i) $0\le\theta_i\le 1/3$ *or* (ii) $2/3\le\theta_i\le 1$ *for all i.*

Remark 2.1. The case in which $\theta_i=0$ for all i, our Theorem 1 correspond to a result proved in Lyche [5] for a uniform mesh. If we assume that θ_i is constant for all i we deduce from Theorem 1 the result contained in Dikshit and Powar [2]. The later result with $h\to 0$ gives a result due to Meir and Sharma [6]. Further Theorem 1 with $h\to 0$ gives the result proved in Chatterjee and Dikshit [1].

Proof of Theorem 1. Using $s\in D_1(3,P,h)$ we get

(2.5) $$\Delta c_i=pM_i;\ \Delta d_i=p(c_i\theta_i^*+c_{i+1}\theta_{i+1})$$

where Δ is the usual foward difference operator. In view of the interpolatory condition (2.3), it follows from (2.4) that

(2.6) $$6f(y_i)=\theta_i^*(\theta_i^{*2}p^2-h^2)M_{i-1}+\theta_i(\theta_i^2p^2-h^2)M_i+6d_i.$$

Writing $b_i=\theta_i^*+\theta_{i+1}$ for all i and combining (2.5)–(2.6), we have

(2.7) $$\bar{R}_iM_{i-1}+\bar{T}_iM_i+T_iM_{i+1}+R_iM_{i+2}=6F_i$$

where $R_i=b_i\theta_{i+2}(\theta_{i+2}^2p^2-h^2)$; $F_i=p^2(b_i\Delta f(y_{i+1})-b_{i+1}\Delta f(y_i))$;

$$T_i=b_i\theta_{i+2}^2(3-\theta_{i+2})p^2+\theta_i^*(((1-\theta_{i+1}^3)+3\theta_{i+2})p^2+(\theta_{i+2}-\theta_{i+1}^*)h^2)$$
$$+\theta_{i+1}(((1-\theta_{i+1}^2)+\theta_{i+2}(3-\theta_{i+1}^2))p^2+2\theta_{i+2}h^2).$$

In order to prove Theorem 1, we shall show that the system of equations (2.7) has a unique set of solutions. It is easily seen that for $h\le p$ and $0\le\theta_i\le 1$, T_i and $\bar{T}_i$ are nonnegative. Also we notice that

$$|R_i|+|\bar{R}_i|\le b_i\theta_{i+2}(\theta_{i+2}^2p^2+h^2)+b_{i+1}\theta_i^*(\theta_i^{*2}p^2+h^2).$$

Writing $J(\theta_i)=1-3\theta_i^2+2\theta_i^3-3\theta_{i+1}^2$, we see that in the coefficient matrix of (2.7) the excess of the positive value of $\bar{T}_i$ over the sum of the positive values of T_i, R_i and $\bar{R}_i$ is not less than

$$t(\theta_i,h)=b_{i+1}((J(\theta_i)+2\theta_{i+1}^3)p^2-2\theta_{i+1}h^2)+b_i(J(\theta_{i+1})p^2+2(\theta_{i+1}^*-\theta_i+2)h^2).$$

Since $1-3\theta_i^2+2\theta_i^3$ is a nonincreasing function of θ for $0\le\theta_i\le 1/3$, we have $1-3\theta_i^2+2\theta_i^3\ge 20/27$ for all i. Again using the hypothesis (i) of Theorem 1, we observe that $J(\theta_i)\ge 11/27$ and $b_i\ge 2/3$ for all i. Thus under the case (i) of Theorem 1 with the hypothesis that $h\le p$, it follows that $t(\theta_i,h)>0$. In the other case in which $2/3\le\theta_i\le 1$, we see that the excess of the positive value of T_i over the sum of the positive values of R_i, $\bar{R}_i$ and $\bar{T}_i$ in (2.7) is not less than $\bar{t}(\theta_i,h)$, which is of course positive. Thus, the coefficient matrix of the system of equations (2.7) is invertible. This completes the proof of Theorem 1.

3. Norm of differences between splines. In this section, we shall compare the discrete periodic cubic spline interpolants of Theorem 1 for $h=u, v>0$. For convenience, we write,

$$(3.1)\qquad (1/t(h))=\max_i \{t(\theta_i, h), \bar{t}(\theta_i h)\}$$

where $t(\theta_i, h)$ is the same as defined in section 2 and $\bar{t}(\theta_i, h)$ is obtained from $t(\theta_i, h)$.

Setting $M_i(u, v)=M_i(u)-M_i(v)$ and $M_i^*(u, v)=u^2M_i(u)-v^2M_i(v)$, we write the single column matrices $(M_i(u, v))$ or $(M_i^*(u, v))$ by $M(u, v)$ or $(M^*(u, v))$. F denotes the single column matrix (F_i). We shall first prove the following preliminary results.

Lemma 3.1. *Let $s(x, h)$ be the unique discrete periodic cubic spline interpolant of f under the assumptions of Theorem* 1. *Then, we have*

$$(3.2)\qquad \|M(u, v)\|\leq 24t(u)t(v)|v^2-u^2|\|F\|$$

and

$$(3.3)\qquad \|M^*(u, v)\|\leq 6K_1t(u)|v^2-u^2|\|F\|$$

where $t(h)$ for $h=u, v$ is given by (3.1) *and $K_1=1+4v^2t(v)$.*

Proof of Lemma 3.1. Let us write the equations (2.7) as

$$(3.4)\qquad A(h)M(h)=6F$$

where $A(h)$ is the coefficient matrix and $M(h)=(M_i(h))$. It follows directly from the equation (3.4) that

$$(3.5)\qquad A(u)M(u, v)=(A(v)-A(u))M(v).$$

However, as already shown in the proof of Theorem 1, $A(h)$ is invertible and its row-max norm, that is

$$(3.6)\qquad \|A^{-1}(h)\|\leq t(h).$$

It may also be seen easily that

$$(3.7)\qquad \|M(v)\|\leq 6t(v)\|F\|$$

and

$$(3.8)\qquad \|A(v)-A(u)\|\leq 4|v^2-u^2|.$$

Thus using the bounds obtained in (3.6)–(3.8), we get (3.2) from (3.5). Similarly starting with the equation (3.4) and following closely the foregoing proof, we prove (3.3). We are now set to prove the following.

Theorem 2. *Suppose $s(x, h)$ is the unique discrete periodic cubic spline interpolant of f under the assumptions of Theorem* 1. *Then for $h=u, v>0$*

$$(3.9)\qquad \|s(x, u)-s(x, v)\|\leq Kt(u)|v^2-u^2|\|F\|$$

where K is some positive function depending on p and v.

Proof of Theorem 2. Evaluating $d_{i+1}-d_i$ from the equation (2.6) and substituting it in (2.5), we determine c_i and d_i. Thus, using these of c_i and d_i in (2.4), we obtain the following representation for the discrete cubic spline interpolant of f for $x\in[x_{i-1}, x_i]$

$$\begin{aligned}6ps(x, h)=&((x_i-x)^{\{3\}}+\bar{R}_i(x-y_i-pb_i)/b_ib_{i+1})M_{i-1}+((x-x_{i-1})^{\{3\}}\\&-(pR_{i-2}/b_{i-2})+(x-y_i)((R_{i-2}/b_{i-2})-(\bar{R}_{i+1}/b_{i+2})-6p^2\theta_{i+1})/b_i)M_i\\&-((x-y_i)R_{i-1}/b_ib_{i-1})M_{i+1}+6pf(y_i)+6(x-y_i)\Delta f(y_i)/b_i.\end{aligned}$$

Thus, setting $x-x_{i-1}=pt$ with $0\leq t\leq 1$, we observe that

$$\begin{aligned}6b_i(s(x,u)-s(x,v))=(\theta_i-t)[(((1-t)^2+\theta_i^{*2}+\theta_i^*(1-t))b_i-\theta_i^{*3})p^2M_{i-1}(u,v)\\+((6\theta_{i+1}+\theta_{i+1}^{*3}-\theta_i^3)-b_i(t^2+\theta_i^2+t\theta_i))p^2M_i(u,v)+\theta_{i+1}^3p^2M_{i+1}(u,v)\\-\theta_{i+1}M_{i-1}^*(u,v)+2\theta_{i+1}M_i^*(u,v)-\theta_{i+1}M_{i+1}^*(u,v)].\end{aligned}$$

Thus,

$$\begin{aligned}6b_i|s(x,u)-s(x,v)|\le p^2(2M_{i-1}(u,v)+4M_i(u,v)+M_{i+1}(u,v))\\+M_{i-1}^*(u,v)+2M_i^*(u,v)+M_{i+1}^*(u,v).\end{aligned}$$

Observing that $b_i\ge 2/3$, we have

$$4\|s(x,u)-s(x,v)\|\le 7p^2\|M(u,v)\|+4\|M^*(u,v)\|. \tag{3.10}$$

Combining (3.10) with the result of Lemma 3.1, we complete the proof of Theorem 2.

4. Discrete error bounds. For a given $h>0$, we define a discrete interval

$$[a,b]_h=\{a+jh\,;\ j=0,1,\cdots,N\}$$

and assume that equidistant mesh points $x_i\in[a,b]_h$, $i=0,1,\cdots,n$. For a function g and three distinct points x_1, x_2, x_3 in its domain the first and second divided differences are defined by

$$[x_1,x_2]g=\{g(x_1)-g(x_2)\}/(x_1-x_2)\quad\text{and}$$
$$[x_1,x_2,x_3]g=\{[x_1,x_2]g-[x_2,x_3]g\}/(x_1-x_3)\quad\text{respectively.}$$

For convenience, we write $D_h^{\{2\}}g=g^{\{2\}}$ and $w(g,p)$ for the modulus of continuity of g. The discrete norm of a function g over the interval $[a,b]_h$ is defined by

$$\|g\|'=\max_{x\in[a,b]_h}|g(x)|.$$

Without assuming any smoothness condition on f, we shall obtain in the following the bounds for the function $e(x)=f(x)-s(x,h)$ over the discrete interval $[a,b]_h$.

Theorem 3. *Let $s(x,h)$ be the unique discrete periodic cubic spline interpolant of f. Then over the discrete interval $[a,b]_h$*

$$\|e^{\{j\}}\|'\le p^{2-j}J(j)K(h)w(f^{\{2\}},p),\quad j=0,1,2 \tag{4.1}$$

where $K(h)$ is a positive function of h given by (4.9) *and $J(0)=1/8$, $J(1)=1/2$ and $J(2)=1$.*

In order to prove Theorem 3, we shall need the following results due to Lyche ([4] Lemma 5.3 and Corollary 5.2 respectively).

Lemma 4.1. *Let $a\equiv\langle a_j\rangle_{j=1}^4$ and $q\equiv\langle q_j\rangle_{j=1}^3$ be given sequence of non-negative real numbers such that $\sum_{j=1}^4 a_j=\sum_{j=1}^3 q_j$. Then for any real valued function f defined on a discrete interval $[\alpha,\beta]_h$, we have*

$$\left|\sum_{j=1}^4 a_j[x_{j0},x_{j1},x_{j2}]f-\sum_{j=1}^3 q_j[y_{j0},y_{j1},y_{j2}]f\right|\le w(f^{\{2\}},|\beta-\alpha-2h|)\sum_{j=1}^3 q_j/2$$

where all the points $x_{j,i}$, $p_{j,i}\in[\alpha,\beta]_h$ for relevant values of i, j.

Lemma 4.2. *Let c, d be given real numbers such that $d=c+rh$ for some positive integer r and for the operators L and R a given function $g:[c-h,d+h]_h\to R$ be such that*

$$p'(Lg)(x)=(x-c)g(d)+(d-x)g(c)\,;\ (Rg)(x)=g(x)-(Lg)(x)$$

where $p'=d-c$. *Then considering the norm over the discrete interval* $[c, d]_h$, *we have*

(4.2) $$\|Rg\|' \leq w(g, p)$$

(4.3) $$\|Rg\|' \leq (p^2/8)\|g^{\{2\}}\|'$$

(4.4) $$\|(Rg)^{\{1\}}\|' \leq (p/2)\|g^{\{2\}}\|'.$$

Proof of Theorem 3. Since all the mesh points $x_i \in [a, b]_h$, we may take for any fixed i, $c=x_{i-1}$, $d=x_i$ in Lemma 4.2. Now taking $g=e^{\{2\}}$, we see that $Re^{\{2\}}=Rf^{\{2\}}$. Also it is clear from the definition of Lg that over the discrete interval $[x_{i-1}-h, x_i+h]_h$

(4.5) $$\|Le^{\{2\}}\|' \leq \max_i |e_i^{\{2\}}|.$$

Now since $e^{\{2\}}=Re^{\{2\}}+Le^{\{2\}}$, we see from (4.5) that over the discrete interval $[x_{i-1}-h, x_i+h]_h$

(4.6) $$\|e^{\{2\}}\|' \leq \|Rf^{\{2\}}\|' + \|(e_i)^{\{2\}}\|'.$$

From (3.4), we obtain the system of equations for $e_i^{\{2\}}$ as follows

(4.7) $$A(h)(e_i^{\{2\}})=A(h)(f_i^{\{2\}})-6(F_i)=(L_i).$$

However, the i-th row of the single column matrix (L_i) may be written as

$$\sum_{j=1}^{4} a_j[x_{j0}, x_{j1}, x_{j2}]f-\sum_{j=1}^{3} q_j[y_{j0}, y_{j1}, y_{j2}]f$$

where $a_1=2b_{i+1}\theta_i^{*3}p^2$, $a_2=2\bar{T}_i$, $a_3=2T_i$, $a_4=2b_i\theta_{i+2}^3p^2$,

$q_1=6p^2b_ib_{i+1}(b_i+b_{i+1})$, $q_2=2b_{i+1}\theta_i^*h^2$, $q_3=2b_i\theta_{i+2}h^2$,

$\langle x_{jk}\rangle_{j=1}^4 \equiv \langle x_{i+j-2}+(k-1)h\rangle$, $\langle y_{1k}\rangle \equiv \langle y_{i+k}\rangle$, $\langle y_{2k}\rangle \equiv \langle x_{i-1}+(k-1)h\rangle$

and $\langle y_{3k}\rangle \equiv \langle x_{i+2}+(k-1)h\rangle$ for $k=0, 1, 2$.

Clearly $\sum_{j=1}^4 a_j = \sum_{j=1}^3 q_j$ and therefore, applying Lemma 4.1 along with the fact $b_i \leq 4/3$, for all i, we have

(4.8) $$|L_i| \leq (16(8p^2+h^2)/3)w(f^{\{2\}}, p).$$

Now using the equations (3.6), (4.2), (4.5)–(4.8), we get

(4.9) $$\|e^{\{2\}}\|' \leq K(h)w(f^{\{2\}}, p)$$

where $K(h)=[1+16t(h)(8p^2+h^2)/3]$.

In order to prove the remaining part of Theorem 3, we take $c=y_i$, $d=y_{i+1}$ in Lemma 4.2 and if $g=e$ then $Re=e$, so that by (4.3), we have

(4.10) $$\|e\|' = \|Re\|' \leq (p^2/8)\|e^{\{2\}}\|'$$

which proves (4.1) for $j=0$. Similarly, we can prove (4.1) for $j=1$ by using (4.4). This completes the proof of Theorem 3.

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