22. *On the Distribution of the Zeros of the Riemann Zeta Function in Short Intervals*

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§1. **Introduction.** Let $\zeta(s)$ be the Riemann zeta function. In [2] and [5], the author has shown that for $T>T_0$, $0<\Delta\ll 1$ and for each integer $k\geq 1$,

$$\int_0^T (S(t+\Delta)-S(t))^{2k}dt$$
$$=\frac{(2k)!}{(2\pi)^{2k}k!}T(2\log(2+\Delta\log T))^k+O(T(\log(2+\Delta\log T))^{k-(1/2)}),$$

where we put $S(t)=(1/\pi)\arg\zeta(1/2+it)$ as usual. This formula has been proved to be powerful in the theory of the Riemann zeta function (cf. [7], [9] and [10], for example). Recently, some attentions have been paid to it from the view point of the comparison with the distribution of the eigenvalues of the Gaussian Unitary Ensembles (cf. [1], [3], [7] and [12]).

In the present article, we shall assume the Riemann Hypothesis and refine the above result for $k=1$ as follows. To state our result we put

$$F(a)\equiv F(a,T)\equiv\left(\frac{T}{2\pi}\log\frac{T}{2\pi}\right)^{-1}\sum_{0<\gamma,\gamma'\leq T}\left(\frac{T}{2\pi}\right)^{ia(\gamma-\gamma')}\frac{4}{4+(\gamma-\gamma')^2},$$

where $a\geq 0$, γ and γ' run over the imaginary parts ($\neq 0$) of the zeros of $\zeta(s)$.

Theorem. *Suppose that* $0<\Delta=o(1)$. *Then we have for* $T>T_0$,

$$\int_0^T (S(t+\Delta)-S(t))^2dt$$
$$=\frac{T}{\pi^2}\left\{\int_0^{\Delta\log(T/2\pi)}\frac{1-\cos(a)}{a}da+\int_1^\infty\frac{F(a)}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\right\}+o(T).$$

We shall prove this by applying Goldston [8] while we have applied Selberg [13] to prove our previous mean value theorem described above.

Some applications of this theorem will be discussed in the forthcoming paper [6].

§2. **Proof of Theorem.** We shall use first the following Goldston's explicit formula for $S(t)$ (cf. p. 157 of [8]). For $t>1, t\neq\gamma$, $x=(T/2\pi)^\beta$ and $0\leq\beta\leq 1$,

$$S(t)=-\frac{1}{\pi}\sum_{n\leq x}\frac{\Lambda(n)\sin(t\log n)}{\sqrt{n}\log n}f\left(\frac{\log n}{\log x}\right)$$
$$+\frac{1}{\pi}\sum_\gamma h((t-\gamma)\log x)+O\left(\frac{1}{t\log^2 x}\right)+O\left(\frac{\sqrt{x}}{t^2\log^2 x}\right)$$
$$=-A(t)+B(t)+O\left(\frac{1}{t\log^2 x}\right)+O\left(\frac{\sqrt{x}}{t^2\log^2 x}\right),\ \text{say},$$

where $\Lambda(x)$ is the von-Mangoldt function,

$$f(u)=\frac{\pi u}{2}\cot\left(\frac{\pi u}{2}\right) \quad\text{and}\quad h(v)=\sin v\int_0^\infty \frac{u}{u^2+v^2}\frac{du}{\sinh u}.$$

Now,

$$\begin{aligned}S&\equiv\int_0^T (S(t+\Delta)-S(t))^2dt\\&=-\int_0^T (A(t+\Delta)-A(t))^2dt-2\int_0^T (S(t+\Delta)-S(t))(A(t+\Delta)-A(t))dt\\&\quad+\int_0^T (B(t+\Delta)-B(t))^2dt+O\left(\frac{x}{\log^4 x}\right)\\&=S_1+S_2+S_3+O\left(\frac{x}{\log^4 x}\right),\text{ say.}\end{aligned}$$

$$\begin{aligned}S_1&=-\frac{1}{\pi^2}\sum_{m,n\le x}\frac{\Lambda(m)\Lambda(n)}{\sqrt{mn}\log m\log n}f\left(\frac{\log m}{\log x}\right)f\left(\frac{\log n}{\log x}\right)\\&\quad\times\int_0^T(\sin((t+\Delta)\log m)-\sin(t\log m))(\sin((t+\Delta)\log n)-\sin(t\log n))dt\\&=-\frac{T}{\pi^2}\sum_{n\le x}\frac{\Lambda^2(n)}{n\log^2 n}f^2\left(\frac{\log n}{\log x}\right)(1-\cos(\Delta\log n))+O\left(x\frac{\log\log x}{\log x}\right).\end{aligned}$$

$$\begin{aligned}S_2&=2\int_0^T S(t)(A(t+\Delta)+A(t-\Delta)-2A(t))dt+O\left(\Delta\frac{\log T}{\log\log T}\frac{\sqrt{x}}{\log x}\right)\\&=\frac{4}{\pi}\sum_{n\le x}\frac{\Lambda(n)}{\sqrt{n}\log n}f\left(\frac{\log n}{\log x}\right)(\cos(\Delta\log n)-1)\\&\quad\times\int_0^T S(t)\sin(t\log n)dt+O\left(\Delta\frac{\log T}{\log\log T}\frac{\sqrt{x}}{\log x}\right)\\&=-\frac{2T}{\pi^2}\sum_{n\le x}\frac{\Lambda^2(n)}{n\log^2 n}f\left(\frac{\log n}{\log x}\right)(\cos(\Delta\log n)-1)+O\left(x\frac{\log\log x}{\log x}\right),\end{aligned}$$

where the last integral is evaluated in [4].

Thus we get

$$\begin{aligned}S_1+S_2&=\frac{T}{\pi^2}\sum_{p\le x}\frac{1}{p}(\cos(\Delta\log p)-1)\left(f^2\left(\frac{\log p}{\log x}\right)-2f\left(\frac{\log p}{\log x}\right)\right)\\&\quad+\frac{T}{\pi^2}\sum_{r=2,p^r\le x}^{\infty}\frac{1}{p^r r^2}(\cos(\Delta\log p^r)-1)\left(f^2\left(\frac{\log p^r}{\log x}\right)-2f\left(\frac{\log p^r}{\log x}\right)\right)\\&\quad+O\left(x\frac{\log\log x}{\log x}\right)\\&=S_4+S_5+O\left(x\frac{\log\log x}{\log x}\right),\text{ say,}\end{aligned}$$

where p runs over the prime numbers.

$$\begin{aligned}S_5&\ll T\left\{\sum_{r=2,p^r\le \mathrm{Min}(x,e^{\sqrt{1/\Delta}})}^{\infty}\frac{\sin^2((\Delta/2)\log p^r)}{p^r r^2}+\sum_{r=2,p^r>e^{\sqrt{1/\Delta}}}^{\infty}\frac{1}{p^r r^2}\right\}\\&=O(T\Delta^2).\end{aligned}$$

Here we notice that with the Euler constant C_0,

$$\sum_{p\le u}\frac{1}{p}=\log\log u+C_0+\sum_p\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)+r(u),$$

$$r(u)\ll\frac{\log u}{\sqrt{u}}\qquad \text{for } u>u_0.$$

Using this we get

$$S_4=\frac{T}{\pi^2}\left\{\int_0^\beta\frac{\cos(a\Delta\log(T/2\pi))-1}{a}\left(\frac{\pi a}{2\beta}\cot\left(\frac{\pi a}{2\beta}\right)-1\right)^2da\right.$$
$$\left.-\int_0^\beta\frac{\cos(a\Delta\log(T/2\pi))-1}{a}da\right\}$$
$$-\frac{T}{\pi^2}\int_0^{\beta\log 2/\log x}\frac{\cos(a\Delta\log(T/2\pi))-1}{a}\left(\left(\frac{\pi a}{2\beta}\right)^2\cot^2\left(\frac{\pi a}{2\beta}\right)-2\frac{\pi a}{2\beta}\cot\left(\frac{\pi a}{2\beta}\right)\right)da$$
$$-\frac{T}{\pi^2}(\cos(\Delta\log 2)-1)\left(f^2\left(\frac{\log 2}{\log x}\right)-2f\left(\frac{\log 2}{\log x}\right)\right)r(2-0)+O\left(\frac{T}{\log x}\right)+O(T\Delta^2)$$
$$=\frac{T}{\pi^2}\left\{\int_0^\beta\frac{\cos(a\Delta\log(T/2\pi))-1}{a}\left(\frac{\pi a}{2\beta}\cot\left(\frac{\pi a}{2\beta}\right)-1\right)^2da\right.$$
$$\left.-\int_0^\beta\frac{\cos(a\Delta\log(T/2\pi))-1}{a}da\right\}+O\left(\frac{T}{\log x}\right)+O(T\Delta^2).$$

We shall next evaluate S_3.

$$S_3=2\int_0^T B^2(t)dt-2\int_0^T B(t+\Delta)B(t)dt+O(\Delta\log^2 T).$$

Using the argument in pp. 158–160 of [8], we get

$$-2\int_0^T B(t+\Delta)B(t)dt=-\frac{2}{\pi^2}\sum_{\gamma,\gamma'}\int_0^T h((t+\Delta-\gamma)\log x)h((t-\gamma')\log x)dt$$
$$=-\frac{2}{\pi^2}\sum_{0<\gamma,\gamma'\le T}\int_{-\infty}^{\infty}h((t+\Delta-\gamma)\log x)h((t-\gamma')\log x)dt+O(\log^3 T)$$
$$=-\frac{2}{\pi^2\log x}\sum_{0<\gamma,\gamma'\le T}\hat{k}((\gamma-\gamma'-\Delta)\log x)+O(\log^3 T),$$

where $k(u)=\left(\frac{1}{2u}-\frac{\pi^2}{2}\cot(\pi^2 u)\right)^2$ for $|u|\le\frac{1}{2\pi}$, and $=\frac{1}{4u^2}$ for $|u|>\frac{1}{2\pi}$ and $\hat{k}(u)$ is the Fourier transform of $k(u)$. Thus we get

$$S_3=\frac{2}{\pi^2\log x}\sum_{0<\gamma,\gamma'\le T}(\hat{k}((\gamma-\gamma')\log x)-\hat{k}((\gamma-\gamma'-\Delta)\log x))+O(\log^3 T).$$

Here we notice that

$$\sum_{0<\gamma,\gamma'\le T}\mathrm{Min}\left(1,\frac{1}{((\gamma-\gamma'-\Delta)\log x)^2}\right)\frac{(\gamma-\gamma')^2}{4+(\gamma-\gamma')^2}$$
$$=\left\{\sum_{\substack{0<\gamma,\gamma'\le T\\ |(\gamma-\gamma'-\Delta)\log x|\ge\Delta\log x}}+\sum_{\substack{0<\gamma,\gamma'\le T\\ |(\gamma-\gamma'-\Delta)\log x|<\Delta\log x}}\right\}\mathrm{Min}\left(1,\frac{1}{((\gamma-\gamma'-\Delta)\log x)^2}\right)\frac{(\gamma-\gamma')^2}{4+(\gamma-\gamma')^2}$$
$$=U_1+U_2,\ \text{say}.$$

$$U_1\ll\frac{1}{\log^2 x}\sum_{0<\gamma,\gamma'\le T}\frac{1}{4+(\gamma-\gamma')^2}\ll T\frac{\log^2 T}{\log^2 x}.$$

$$U_2\ll\Delta^2\sum_{\substack{0<\gamma,\gamma'\le T\\ |\gamma-\gamma'-\Delta|\le 1/\log x}}\cdot 1+\Delta^2\sum_{1\le n<<\Delta\log x}\frac{1}{n^2}\sum_{\substack{0<\gamma,\gamma'\le T\\ n/\log x<|\gamma-\gamma'-\Delta|\le(n+1)/\log x}}\cdot 1$$
$$=U_3+U_4,\ \text{say},$$

where we suppose that $U_4=0$ if $\Delta \log x \leq 1$. To estimate the last two terms we apply the following lemma.

Lemma. *Suppose that $a \ll T^A$ with some positive constant A. Then we have*

$$\sum_{0<\gamma\leq T, \gamma+a>0} S(\gamma+a) \ll T \log T.$$

To prove this, we use Selberg's explicit formula for $S(t)$ [13] and the author's theorem in [4]. The details will appear in [6].

Using this we get

$$U_3 \ll \Delta^2 \sum_{0<\gamma\leq T} \sum_{\gamma-\Delta-1/\log x \leq \gamma' \leq \gamma-\Delta+1/\log x} \cdot 1$$
$$\ll \Delta^2 T \frac{\log^2 T}{\log x} + \Delta^2 \left| \sum_{0<\gamma\leq T} S\left(\gamma-\Delta+\frac{1}{\log x}\right)\right| + \Delta^2 \left| \sum_{0<\gamma\leq T} S\left(\gamma-\Delta-\frac{1}{\log x}\right)\right|$$
$$\ll \Delta^2 T \log \mathrm{T}.$$
$$U_4 \ll \Delta^2 T \frac{\log^2 T}{\log x} + \Delta^2 \sum_{1\leq n < < \Delta \log x} \frac{1}{n^2} \left| \sum_{0<\gamma\leq T} S\left(\gamma-\Delta+\frac{n}{\log x}\right)\right|$$
$$+ \Delta^2 \sum_{1\leq n < < \Delta \log x} \frac{1}{n^2} \left| \sum_{0<\gamma\leq T} S\left(\gamma-\Delta-\frac{n}{\log x}\right)\right|$$
$$\ll \Delta^2 T \log T.$$

Thus by p. 162 of [8], we get

$$S_3 = \frac{T \log (T/2\pi)}{\pi^4 \beta \log x} \int_0^\infty F(a) k\left(\frac{a}{2\pi\beta}\right)\left(1-\cos\left(a\Delta \log \frac{T}{2\pi}\right)\right) da + O\left(\frac{T}{\log T}\right) + O(T\Delta^2)$$
$$= \frac{T}{\pi^2}\left\{\int_0^\beta \frac{1}{a}\left(1-\frac{\pi a}{2\beta}\cot\left(\frac{\pi a}{2\beta}\right)\right)^2 \left(1-\cos\left(a\Delta \log \frac{T}{2\pi}\right)\right) da\right.$$
$$\left. + \int_\beta^1 \frac{(1-\cos(a\Delta \log (T/2\pi)))}{a} da + \int_1^\infty \frac{F(a)}{a^2}\left(1-\cos\left(a\Delta \log \frac{T}{2\pi}\right)\right) da\right\}$$
$$+ O\left(T\left(\Delta^2 + \sqrt{\frac{\log\log T}{\log T}}\right)\right),$$

where we use Montgomery's ([11]) and Goldston-Montgomery's ([9]) results on $F(a)$ for $0 \leq a \leq 1$, in particular, Lemma 8 of [9].

Combining all of our evaluations, we get

$$S = \frac{T}{\pi^2}\left\{\int_0^{\Delta \log (T/2\pi)} \frac{1-\cos(a)}{a} da + \int_1^\infty \frac{F(a)}{a^2}\left(1-\cos\left(a\Delta \log \frac{T}{2\pi}\right)\right) da\right\}$$
$$+ O\left(T\left(\Delta^2 + \sqrt{\frac{\log\log T}{\log T}}\right)\right).$$

This implies our theorem as described in the introduction.

§ 3. Concluding remarks.

3-1. $$\int_0^{\Delta \log (T/2\pi)} \frac{1-\cos(a)}{a} da = \log\left(\Delta \log \frac{T}{2\pi}\right) + C_0 - Ci\left(\Delta \log \frac{T}{2\pi}\right).$$

By Theorem 2 of Goldston [8], we have

$$\left| \int_1^\infty \frac{F(a)}{a^2}\left(1-\cos\left(a\Delta \log \frac{T}{2\pi}\right)\right) da \right| \leq 2 \int_1^\infty \frac{F(a)}{a^2} da < 4.$$

3-2. If we assume also Montgomery's conjecture [11] on $F(a)$, then we have

$$\int_1^\infty \frac{F(a)}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da$$
$$=\int_1^\infty \frac{1}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da+o(1)$$
$$=1-\cos\left(\Delta\log\frac{T}{2\pi}\right)+\frac{\pi}{2}\Delta\log\frac{T}{2\pi}-\Delta\log\frac{T}{2\pi}\cdot Si\left(\Delta\log\frac{T}{2\pi}\right)+o(1).$$

Thus putting $\Delta=\frac{2\pi\alpha}{\log(T/2\pi)}$, we get for $0<\alpha=o(\log T)$,

$$\int_0^T\left(S\left(t+\frac{2\pi\alpha}{\log(T/2\pi)}\right)-S(t)\right)^2dt$$
$$=\frac{T}{\pi^2}\{\log(2\pi\alpha)+C_0-Ci(2\pi\alpha)+1-\cos(2\pi\alpha)+\pi^2\alpha-2\pi\alpha Si(2\pi\alpha)+o(1)\}.$$

In fact, for any $\alpha>0$ the right hand side is nothing but the GUE part of Berry's formula (19) conjectured in [1] (cf. also [3], [7] and Odlyzko's various comments on this in [12]).

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References

[1] M. V. Berry: Semiclassical formula for the number variance of the Riemann zeros. Nonlinearity, **1**, 399–407 (1988).

[2] A. Fujii: On the distribution of the zeros of the Riemann zeta function in short intervals. Bull. of A.M.S., **81**, no. 1, 139–142 (1975).

[3] ——: Gram's law for the zeta zeros and the eigenvalues of Gaussian unitary ensembles. Proc. Japan Acad., **63**A, 392–395 (1987).

[4] ——: On a theorem of Landau. ibid., **65**A, 51–54 (1989).

[5] ——: On the zeros of Dirichlet L-functions. I. Trans. of A.M.S., **196**, 225–235 (1974); **267**, 33–40 (1981).

[6] ——: On the gaps between the consecutive zeros of the Riemann zeta function (to appear).

[7] P. X. Gallagher and J. Mueller: Primes and zeros in short intervals. Crelle J., **303**, 205–220 (1978).

[8] D. A. Goldston: On the function $S(t)$ in the theory of the Riemann zeta function. J. Number Theory, **27**, 149–177 (1987).

[9] D. A. Goldston and H. L. Montgomery: Pair correlation of zeros and primes in short intervals. Analytic Number Theory and Diophantine Problems, Birkhauser, pp. 183–203 (1987).

[10] D. A. Hejhal: On the distribution of $\log|\zeta'(\frac{1}{2}+it)|$. Number Theory, Trace Formulas and Discrete Groups. Academic Press, pp. 343–370 (1989).

[11] H. L. Montgomery: The pair correlation of the zeros of the zeta function. Proc. Symp. Pure Math., **24**, A.M.S., 181–193 (1973).

[12] A. M. Odlyzko: The 10^{20}-th zero of the Riemann zeta function and 70 million of its neighbours (preprint).

[13] A. Selberg: Contributions to the theory of the Riemann zeta function. Arch. Math. Naturvid., **48**, 89–155 (1946).