21. Nonlinear Singular First Order Partial Differential Equations of Briot-Bouquet Type

By Raymond GÉRARD*) and Hidetoshi TAHARA**)

(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1990)

In this paper we will present a generalization of the Briot-Bouquet ordinary differential equation to partial differential equations.

§1. **Briot-Bouquet equation.** First let us recall the theory of nonlinear ordinary differential equations of the form

$$t\frac{du}{dt}=f(t,u),\quad f(0,0)=0 \tag{1.1}$$

which was first studied by Briot-Bouquet [1]. Nowadays it is called the Briot-Bouquet equation and the structure of solutions of (1.1) near the origin of C_t is well-known (see Hille [3], Hukuhara-Kimura-Matuda [4], Kimura [5], Gérard [2] etc.). In particular, when

$$\rho=\frac{\partial f}{\partial u}(0,0)$$

is in a generic position, we know the following:

Theorem 1. *Assume that $f(t,u)$ is a holomorphic function defined near the origin of $C_t\times C_u$. Then we have:*

(1) (*Holomorphic solutions*). *If $\rho\notin N^*(=\{1,2,3,\cdots\})$, the equation* (1.1) *has a unique solution $u_0(t)$ holomorphic near the origin of C_t satisfying $u_0(0)=0$.*

(2) (*Singular solutions*). *If $\rho\notin N^*\cup\{a\in R;\, a\leqq 0\}$, the general solution $u(t)$ of* (1.1) *near the origin of C_t is given by*

$$u(t)=ct^{\rho}+a_{1,0}t+\sum_{i+j\geqq 2}a_{i,j}t^i(ct^{\rho})^j, \tag{1.2}$$

where $c\in C$ is arbitrary, the coefficients $a_{i,j}\in C$ are uniquely determined by the equation (1.1), *and the series*

$$w+a_{1,0}t+\sum_{i+j\geqq 2}a_{i,j}t^iw^j$$

is a convergent power series in $\{t,w\}$. The holomorphic solution $u_0(t)$ in (1) *is given by the case $c=0$.*

§2. **Generalization of (1.1) to partial differential equations.** Let us consider

$$t\frac{\partial u}{\partial t}=F\Big(t,x,u,\frac{\partial u}{\partial x}\Big), \tag{2.1}$$

where $(t,x)\in C_t\times C_x^n$, $x=(x_1,\cdots,x_n)$, $\dfrac{\partial u}{\partial x}=\Big(\dfrac{\partial u}{\partial x_1},\cdots,\dfrac{\partial u}{\partial x_n}\Big)$ and $F(t,x,u,v)$ with $v=(v_1,\cdots,v_n)$ is a function defined in a polydisk Δ centered at the

*) Institut de Recherche Mathématique Alsacien, Université Louis Pasteur.
**) Department of Mathematics, Sophia University.

origin of $C_t\times C_x^n\times C_u\times C_v^n$. Put $\Delta_0=\Delta\cap\{t=0, u=0 \text{ and } v=0\}$. Assume:

(A_1) $F(t, x, u, v)$ is holomorphic in Δ,

(A_2) $F(0, x, 0, 0)=0$ in Δ_0,

(A_3) $\dfrac{\partial F}{\partial v_i}(0, x, 0, 0)=0$ in Δ_0 for $i=1, \cdots, n$.

Then we will propose here the following definition.

Definition. If (2.1) satisfies (A_1), (A_2) and (A_3), we say that (2.1) is a partial differential equation *of Briot-Bouquet type* with respect to t.

The reasonability of this definition will be seen in Theorem 2 given in § 3.

§ 3. Main result for (2.1). Denote by:

– $\widetilde{C\setminus\{0\}}$ the universal covering space of $C\setminus\{0\}$;

– S_θ the sector in $\widetilde{C\setminus\{0\}}$ defined by $\{t\in\widetilde{C\setminus\{0\}};\ |\arg t|<\theta\}$;

– $S(\varepsilon(s))=\{t\in\widetilde{C\setminus\{0\}};\ 0<|t|<\varepsilon(\arg t)\}$ for some positive-valued function $\varepsilon(s)$ defined and continuous on R_s;

– $D(\delta)=\{x\in C^n;\ |x_i|<\delta,\ i=1, \cdots, n\}$;

– $C\{x\}$ the ring of germs of holomorphic functions at the origin of C_x^n;

– $\tilde{\mathcal{O}}_+$ the set of all functions $u(t, x)$ satisfying the following conditions (i) and (ii):

(i) $u(t, x)$ is holomorphic in $S(\varepsilon(s))\times D(\delta)$ for some $\varepsilon(s)$ and $\delta>0$;

(ii) there is an $a>0$ such that for any $\theta>0$ and any compact subset K of $D(\delta)$

$$\max_{x\in K}|u(t, x)|=O(|t|^a)$$

as t tends to zero in S_θ.

Put

$$\rho(x)=\frac{\partial F}{\partial u}(0, x, 0, 0). \tag{3.1}$$

Then the main result for (2.1) is stated as follows:

Theorem 2. *Assume* (A_1), (A_2), (A_3) *and* $\rho(0)\notin N^*$. *Then we have*:

(1) (*Holomorphic solutions*). *The equation* (2.1) *has a unique solution* $u_0(t, x)$ *holomorphic near the origin of* $C_t\times C_x^n$ *satisfying* $u_0(0, x)\equiv 0$.

(2) (*Singular solutions*). *Denote by* $\mathcal{S}_+$ *the set of all* $\tilde{\mathcal{O}}_+$*-solutions of* (2.1). *Then*:

$$\mathcal{S}_+=\begin{cases}\{u_0\}, & \textit{when } \operatorname{Re}\rho(0)\leqq 0,\\ \{u_0\}\cup\{U(\varphi);\ 0\not\equiv\varphi(x)\in C\{x\}\}, & \textit{when } \operatorname{Re}\rho(0)>0,\end{cases}$$

where u_0 *is the holomorphic solution in* (1), *and* $U(\varphi)$ *is an* $\tilde{\mathcal{O}}_+$*-solution of* (2.1) *having the expansion of the following form*

$$\begin{cases}U(\varphi)=\sum_{i\geqq 1}u_i(x)t^i+\sum_{\substack{i+2j\geqq k+2\\ j\geqq 1}}\varphi_{i,j,k}(x)t^{i+j\rho(x)}(\log t)^k,\\ \varphi_{0,1,0}(x)=\varphi(x).\end{cases}$$

Remark 1. Note that the above result is consistent with the one for (1.1). To see this, we have only to recall that (1.1) is transformed into the equation $\left(t\dfrac{\partial}{\partial t}-\rho\right)w=0$ under the relation

$$u=w+a_{1,0}t+\sum_{i+j\geqq 2}a_{i,j}t^i w^j$$

and therefore in (1.2) the condition $u(t)\in\tilde{\mathcal{O}}_+$ is equivalent to the condition $ct^\rho \in \tilde{\mathcal{O}}_+$ (see Hukuhara-Kimura-Matuda [4]).

Remark 2. Let us consider

$$t\frac{\partial u}{\partial t}=G\left(t, x, u, t\frac{\partial u}{\partial x}\right), \tag{3.2}$$

where $G(t, x, u, v)$ is a holomorphic function in Δ satisfying $G(0, x, 0, 0)=0$ in Δ_0. Then the equation (3.2) is a particular form of partial differential equations of Briot-Bouquet type with respect to t in our sense. In this case, the $\tilde{\mathcal{O}}_+$-solution $U(\varphi)$ in Theorem 2 is reduced to a solution of the following form:

$$\begin{cases} U(\varphi)=\sum_{i+j+k\geqq 1}\varphi_{i,j,k}(x)t^i(t^{\rho(x)})^j(t\log t)^k, \\ \varphi_{0,1,0}(x)=\varphi(x). \end{cases}$$

Thus, our equation (2.1) is quite similar to the Briot-Bouquet equation (1.1) not only in the form of the equation but also in the structure of solutions and therefore the definition in §2 will be reasonable. The holomorphic function $\rho(x)$ defined by (3.1) may be called the characteristic exponent function of (2.1).

Details and proofs will be published elsewhere.

References

[1] Ch. Briot and J. Cl. Bouquet: Recherches sur les propriétés des fonctions définies par des équations différentielles. J. Ecole Polytech., **21**, 133–197 (1856).

[2] R. Gérard: Etude locale des équations différentielles de la forme $xy'=f(x, y)$ au voisinage de $x=0$. J. Fac. Sci. Univ. Tokyo, **36** (1989).

[3] E. Hille: Ordinary Differential Equations in the Complex Domain. John Wiley and Sons (1976).

[4] M. Hukuhara, T. Kimura, and T. Matuda: Equations différentielles ordinaires du premier order dans le champ complexe. Publ. of the Math. Soc. of Japan (1961).

[5] T. Kimura: Ordinary Differential Equations. Iwanami Shoten (1977) (in Japanese).