20. Minimal Currents and Relaxation of Variational Integrals on Mappings of Bounded Variation

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1. Introduction and main results. Let $T$ be a 1-dimensional current of locally finite mass on $\mathbb{R}^n$. By the Riesz representation theorem $T$ is identified with a $\mathbb{R}^n$-valued Radon measure $T=(T^1, \ldots, T^n)$ on $\mathbb{R}^n$ (see e.g., [5,12]). If $F=F(y, \eta)$ is a nonnegative continuous function on $\mathbb{R}^n \times \mathbb{R}^n$ and is positively homogeneous of degree one in $\eta$, a new measure $F(y, T)$ is associated with $T$ (cf. [10]). We consider a functional

$$I_\rho(T) = \int_{\mathbb{R}^m} F(y, T).$$

Here $F$ is assumed to be convex in $\eta$ and satisfy a growth condition

$$k|\eta| \leq F(y, \eta) \leq K|\eta|$$

with $K \geq k > 0$ independent of $y$ and $\eta$. If $T$ is a current representing an oriented $C^1$ curve $C$, $I_\rho(T)$ is the length of the curve $C$ with metric density $F$, so $I_\rho(T)$ agrees with the standard length of $C$ in $\mathbb{R}^n$ when $F(y, \eta) = |\eta|$.

We call $S$ a minimal current from $a \in \mathbb{R}^n$ to $b \in \mathbb{R}^n$ if

$$I_\rho(S) = \inf \{I_\rho(T) ; T \in \mathcal{H}_k, \partial T = \delta_a - \delta_b \}.$$

Here $\delta_a$ denotes the Dirac measure supported at $a$ and $\partial T$ denotes the boundary of $T$, i.e. $\partial T = \text{div} T$. The space $\mathcal{H}$ represents the set of all 1-currents of locally finite mass in $\mathbb{R}^n$. Our main result on minimal currents asserts that a shortest curve is a minimal current.

Theorem 1. There exists a current representing a simple Lipschitz curve from $a$ to $b$ which is a minimal current. In particular,

$$\inf_{\mathcal{H}} I_\rho(T) = \inf \left\{ \int_0^1 F(\gamma(t), \dot{\gamma}(t))dt ; \gamma : [0, 1] \rightarrow \mathbb{R}^m \right\}$$

is Lipschitz and $\gamma(0) = a, \gamma(1) = b \} \quad (\dot{\gamma} = d\gamma/dt).$

If $F(y, \eta)$ is independent of $y$, we have proved in [2, Lemma 8.3] that the straight line from $a$ to $b$ is a minimal current. Theorem 1 has important applications in relaxations of variational integrals on $BV(\Omega, \mathbb{R}^n)$, the set of mapping $u : \Omega \rightarrow \mathbb{R}^m$ of bounded variation, where $\Omega$ is an open set in $\mathbb{R}^n$.

We consider a functional $\mathcal{D}$ of $C^1$ mapping $u : \Omega \rightarrow \mathbb{R}^m$

$$\mathcal{D}(u) = \int_\Omega f(x, u(x), F u(x))dx.$$ 

The density function $f=f(x, y, \xi)$ we discuss here is a nonnegative continuous function in $\Omega \times \mathbb{R}^m \times \mathbb{R}^m$ and convex in $\xi$. Here the Jacobi matrix $F u(x)$ of $u$ at $x$ is identified with an element of $\mathbb{R}^m$. We do not assume homoge-

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nuity but a growth condition

\[ k|\xi| \leq f(x, y, \xi) \leq K(|\xi| + 1). \]

Under these conditions it is well-known that the recession function

\[ f_\infty(x, y, \xi) = \lim_{t \to 0} f(x, y, \xi/t) \]

exists and has the homogeneity in \( \xi \) as well as all other properties of \( f \). For technical reasons we further assume the following equicontinuity. For every \((x_0, y_0) \in \Omega \times \mathbb{R}^m\) and \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( |x - x_0|, |y - y_0| < \delta \) implies

\[ |f(x, y, \xi) - f(x_0, y_0, \xi)| \leq \varepsilon (1 + |\xi|). \]

Let \( \overline{\mathcal{F}} \) be the lower semicontinuous \( L^1_{\text{loc}} \) relaxation of \( \mathcal{F} \) on \( BV(\Omega, \mathbb{R}) \), that is

\[ \overline{\mathcal{F}}(u) = \inf \{ \lim_{i \to \infty} \mathcal{F}(u_i); \ u_i \to u \text{ in } L^1_{\text{loc}}(\Omega, \mathbb{R}) \text{ and } u_i \text{ is } C^1 \}. \]

Our problem is to find an explicit representation of \( \overline{\mathcal{F}} \) for \( u \in BV(\Omega, \mathbb{R}) \). This question is posed by De Giorgi [4]. When \( f \) does not depend on \( y \) this problem is solved by [6, 8, 10]. If \( f \) depends on \( y \), so far only the cases \( m = 1 \) and \( n = 1 \) were settled by [3] and [11], respectively.

We shall answer to this problem for arbitrary \( n, m \geq 1 \) assuming that \( f \) satisfies an isotropy condition

\[ f(x, y, (\xi_i)) \geq f(x, y, \left( \sum_{i=1}^{n} q_i \xi_i \right)), \]

where \( q = (q_1, \ldots, q_n) \in \mathbb{R}^n \) and \( \xi = (\xi_i) \in \mathbb{R}^m \), \( 1 \leq i \leq n, 1 \leq j \leq m \). For \( u \in BV(\Omega, \mathbb{R}) \) it is well-known [5, 7, 12] that \( \nabla u \) is a (matrix) Radon measure decomposed as

\[ \nabla u = \nabla u|_{\Omega_0} + \nabla u|_{(\Omega - \Omega_0 - \Sigma)} + \nu \otimes (u^+ - u^-) \mathcal{H}^{n-1}|_{\Sigma}. \]

Here \( \Sigma \) denotes the set of jump discontinuities of \( u \) and \( \nu \) represents a unit normal to \( \Sigma \). The functions \( u^\pm \) are the trace of \( u \) on \( \Sigma \) defined by \( u^\pm(x) = \lim_{\tau \to \pm 0} u(x \pm \tau \nu(x)) \) and \( \mathcal{H}^{n-1} \) denotes the \( n-1 \) dimensional Hausdorff measure.

By \( \mu|_A \) we mean a measure on \( \Omega \) defined by \( (\mu|_A)(B) = \mu(A \cap B) \) for \( B \subset \Omega \), where \( \mu \) is a measure. For \( a, b \in \mathbb{R}^m \) and \( q \in \mathbb{R}^n \) we introduce a distance like function:

\[ D_q(a, b, q) = \inf \left\{ \int_0^1 f_\infty(x, \gamma(t), q \otimes \gamma'(t)) dt; \right\} \]

(5)

\[ \gamma : [0, 1] \to \mathbb{R}^n \] is Lipschitz and \( \gamma(0) = a, \gamma(1) = b \)

A combination of Theorem 1 and results in [2] yield our main result for relaxation of \( \mathcal{F} \) when \( f \) satisfies all above assumptions. By \( |\mu| \) we mean the total variation measure of \( \mu \) and \( d\mu/d|\mu| \) denotes the Radon-Nikodym derivative.

Theorem 2. For \( u \in BV(\Omega, \mathbb{R}) \) it holds

\[ \overline{\mathcal{F}}(u) = \int_{\Omega_0} f(x, u(x), \nabla u(x)) dx + \int_{\Omega - \Omega_0} f_\infty(x, u(x), \frac{d\nabla u}{d|\nabla u|}(x))|\nabla u| \]

(6)

\[ + \int_x D_q(u^+(x), u^-(x), \nu(x)) d\mathcal{H}^{n-1}(x). \]

*** This terminology is due to De Giorgi [4]. It is also called the lower semicontinuous envelope.
In this note we just give a brief sketch of proofs; the details will be published elsewhere.

After this work is completed, we are informed of a recent work of Ambrosio, Mortola and Tortorelli [1] which proves only "\( \geq \)" in (6) of Theorem 2 without (4). Moreover, they show that equality in (6) does not necessarily hold without assuming (4).

2. Discretization and networks. We approximate a current connecting \( a \) and \( b \) by real polyhedral chain (see [5] for the definition).

Lemma 3. Suppose that \( T \in \mathcal{M}_0 \) satisfies \( \partial T = \delta_b - \delta_a \), \( a, b \in \mathbb{R}^n \) and that its total mass \( M(T) \) is finite. There is a sequence of real polyhedral chain \( T_n \in \mathcal{M}_0 \) with \( \partial T_n = \delta_b - \delta_a \) such that \( T_n \) converges weakly to \( T \) and that \( M(T_n) \rightarrow M(T) \) as \( \epsilon \rightarrow 0 \).

Sketch of the proof. We take \( L \in \mathcal{M}_0 \) representing a piecewise linear curve from \( a \) to \( b \) such that \( M(T) = M(L) + M(R) \) with \( R = T - L \). We may assume that \( R \) is smooth by a standard mollification. Since \( \partial R = 0 \), Poincaré’s lemma implies that there is a smooth 2-current \( \Phi \) such that \( R = \partial \Phi \). We next approximate \( \Phi \) by a piecewise linear \( \Psi \) with compact support associated with a simplicial decomposition of a large cube. For simplicity we only discuss the case \( m = 2 \) so that \( \Psi \) is a scalar function. We approximate \( \Psi \) by a piecewise constant function

\[
\Psi(x) = k, \quad \text{if } (k+1)\varepsilon \leq \Psi(x) < k\varepsilon, \quad k: \text{integer}
\]

so that \( \Psi \rightarrow \Psi \) and \( M(\partial \Psi) \rightarrow M(\partial \Psi) \) as \( \varepsilon \rightarrow 0 \). We take \( \theta \in \mathbb{R} \) such that

\[
M(\Psi + \partial \Psi) = M(L) + M(\partial \Psi).
\]

We thus find a desired approximation \( T = L + \partial \Psi \).

Sketch of the proof of Theorem 1. Let \( \{T_j\} \) be a minimizing sequence of (7)

\[ \inf \{ I_{\psi}(T); T \in \mathcal{M}_0, \partial T = \delta_b - \delta_a \}. \]

By Lemma 3 we approximate \( T_j \) by a real polyhedral chain \( T_{j, n} \). Let \( P \) denote the support of \( T_{j,n} \). Since \( P \) is regarded as a network, applying the theory of minimal flow problem (see e.g. [9]) to

\[ \inf \{ I_{\psi}(T); T \text{ is real polyhedral chain supported in } P \text{ and } \partial T = \delta_b - \delta_a \} \]

we see the infimum is attained at multiplicity one current \( S_{j,n} \), representing a Lipschitz curve from \( a \) to \( b \). By Reshetnyak’s continuity theorem [10] \( M(T_{j,n}) \rightarrow M(T_j) \) with (2) implies \( I_{\psi}(T_{j,n}) \rightarrow I_{\psi}(T_j) \) as \( \epsilon \rightarrow 0 \). We now observe that \( S_{j,n} \) is a minimizing sequence of (7) by taking a subsequence \( \epsilon = \varepsilon_j \rightarrow 0 \) since \( I_{\psi}(S_{j,n}) \leq I_{\psi}(T_{j,n}) \). This proves (3). By a standard compactness argument and (2) we see the infimum of the right hand side of (3) is attained at a simple Lipschitz curve from \( a \) to \( b \).

3. Sketch of the proof of Theorem 2. We shall prove "\( \geq \)" in (6). Let \( \bar{D}_n \) denote

\[ \bar{D}_n(a, b, \nu) = \inf \left\{ \int_{\mathbb{R}^n} f(x, y, (S_i)); \partial S_i = \nu(S_i - \delta_a), S_i \in \mathcal{M}_0, 1 \leq i \leq n \right\}. \]

From main results in [2, Theorems 5.1 and 8.1] it follows that

\[ \overline{\mathcal{F}}(u) = \int_{\partial \mathcal{H}} f(x, u, \mathcal{H}u) dx + \int_{\partial \mathcal{H} - \mathcal{H}^-} f(x, u, \frac{\partial u}{\partial |\mathcal{H}u|}) |\mathcal{H}u| + \int_{\mathcal{H}^-} \theta(x) d\mathcal{H}^{n-1}(x) \]
with some \( \theta \) satisfying \( \theta(x) \geq \tilde{D}_x(u^-(x), u^+(x), \nu(x)) \).

Applying (4) and Theorem 1 with \( F(y, \eta) = f_\omega(x, y, \nu(x) \otimes \eta) \) in (1) yields
\[
\tilde{D}_x(a, b, \nu) \geq D_\omega(a, b, \nu),
\]
where \( D_\omega \) is defined in (5). We thus prove \( \geq \) in (6).

The converse inequality is proved by approximating \( u \) by piecewise constant functions. We note that this part is independently proved by [1].

References


