19. Certain Quadratic First Integral and Elliptic Orbits of Linear Hamiltonian System

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1. Introduction. This paper deals with a close relation between a hyperplane filled with elliptic orbits of a linear Hamiltonian system and a certain quadratic first integral. To be more precise, it is proved that when a linear Hamiltonian system admits an invariant hyperplane filled with closed orbits, it leaves a quadratic form invariant, and conversely, when a certain quadratic first integral is admitted, there exists such an invariant hyperplane.

By the way, the phase portrait drawn by a discrete-time system which approximates a continuous Hamiltonian system is often different from that of the original system. For example, a closed orbit of the original system is usually destroyed by a discrete system, even when the original one is linear. It seems that the result of this paper is of use for the purpose of reproducing the original elliptic orbit by a discrete system when a certain kind of first integrals is inherited.

2. Elliptic orbit of linear system. Let us think of a linear Hamiltonian system with N degrees of freedom given by

$$\frac{dx}{dt}=Hx, \quad H\in sp(N,R), \quad x\in R^{2N}. \tag{1}$$

We introduce into the phase space R^{2N} both a Euclidean inner product $(x, y) = {}^t xy$ and a symplectic inner product $\langle x, y\rangle = {}^t xJy$, where $J=\begin{bmatrix} 0 & I \\ -I & 0\end{bmatrix}$ and the superfix t denotes matrix transpose. An orbit of (1) which starts from x_0 is closed, it and only if $e^{\varepsilon H}x_0=x_0$ holds for a positive constant ε. This condition is equivalent to that H has pure imaginary eigenvalues, in other words, H^2 has a negative eigenvalue. Then, we define a linear subspace by

$$\Gamma_\beta=\{x\in R^{2N} \mid H^2x=-\beta^2x\} \qquad (\beta>0), \tag{2}$$

and assume that $\Gamma_\beta\neq\{0\}$ from now on. Let us pay attention to the solution curves of (1) which are contained in Γ_β. Choose an arbitrary $q\in H_\beta$, $q\neq 0$, and put

$$p=-\frac{1}{\beta}Hq. \tag{3}$$

Then, q and p are linearly independent and spans a two-dimensional hyperplane Γ included by Γ_β.

Proposition 1. *The orbit of* (1) *starting from* $q\in\Gamma$ *is an ellipse with the period* $2\pi/\beta$, *and lies in* Γ. *Furthermore, all elliptic orbits in* Γ *are* *similar to each other.*

Proof. Denote by $q(t)$ and $p(t)$ the solutions of (1) starting from q and p, respectively. Since it holds that

$$(4) \qquad Hq=-\beta p, \qquad Hp=\beta q,$$

we have $(q(t), p(t))=e^{tH}(q, p)=(q, p)\begin{bmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{bmatrix}$. This shows that the curve $q(t)$ is a closed quadratic one in Γ.

The result means that Γ_β is full of ellipses with the period $2\pi/\beta$. A similar circumstance holds true in any other eigenspace Γ_γ of H^2 when the eigenvalue is negative. In general, an orbit starting from a point in the sum $\Gamma_\beta+\Gamma_\gamma$ is, however, like a Lissajous figure, and may not be closed.

We have not used the assumption that H belongs to $sp(N, R)$, and the result holds good in every linear system. In case of linear Hamiltonian systems, the above plane Γ is characterized by a certain quadratic first integral, which is the theme of the next section.

3. Certain quadratic first integral. In this section, we show a close relation between a certain first integral and the hyperplane Γ. Let Γ, Γ_β, q and p be as in § 1. Furthermore, we remark that

$$(5) \qquad {}^tHJ+JH=0$$

holds, for H belongs to $sp(N, R)$.

Theorem 2. *For an arbitrary $q\in\Gamma$ $(q\neq 0)$, define p by* (3). *Then, the following quadratic form is a first integral of* (1)

$$(6) \qquad I(x)=\frac{1}{2}{}^tx{}^tJ(q{}^tq+p{}^tp)Jx.$$

Proof. Put

(7.a) $S=q{}^tq+p{}^tp$, (7.b) $\tilde{S}={}^tJSJ$,

and use (5), and we have $dI/dt={}^txJ(HS+S{}^tH)Jx/2$. Since $HS=-\beta p{}^tq+\beta q{}^tp=-(p{}^t(Hp)+q{}^t(Hq))=-S{}^tH$, then dI/dt vanishes.

Remark. The first integral (6) is independent of the choice of $q\in\Gamma$. In fact, a quadratic form constructed from another $\tilde{q}\in\Gamma$ in the same way becomes (6) multiplied by a constant.

We show several properties of S:

(8.a) $\operatorname{rank}(S)=2$, (8.b) $S\geq 0$,

$\operatorname{range}(S)=\Gamma$, and $\ker(S)=\Gamma^\perp$, where $\Gamma^\perp$ denotes the orthogonal complement of Γ with respect to $(\ ,\ )$ and $S\geq 0$ means that S is non-negative definite. These are restated in terms of $\tilde{S}$ as follows: $\operatorname{rank}(\tilde{S})=2$, $\tilde{S}\geq 0$, $\operatorname{range}(\tilde{S})=J(\Gamma)$, and $\ker(\tilde{S})=J(\Gamma^\perp)$. Now, the next step is to show that the converse of the theorem is true if a slight condition is added. We give a definition concerning the condition.

Definition 1. A linear subspace Ξ is called *null* [1], if and only if $\langle q, p\rangle$ vanishes for all vectors q and p in Ξ.

As for Γ in the above, when the eigenvalues $\pm i\beta$ of H are simple, it is not null. When Γ is not null, $\langle q, p\rangle$ is not zero for arbitrary linearly independent elements q and p. Furthermore, Γ is null if and only if $\Gamma\subset J(\Gamma^\perp)$.

Theorem 3. *Suppose that* (1) *has a quadratic first integral given by* $I(x)={}^t x\tilde{S}x/2$ *subject to* ${}^t\tilde{S}=\tilde{S}$, $\mathrm{rank}(\tilde{S})=2$, *and* $\tilde{S}\geq 0$. *If* $\Gamma=\mathrm{range}(J\tilde{S})$ *is not null and* $H(\Gamma)\neq\{0\}$, *then* Γ *is filled with elliptic orbits of* (1) *with a single period.*

Proof. If we define S by (7.b), then S satisfies (8). Therefore, S is expressed as (7.a) in terms of two non-zero vectors q and p subject to $(q,p)=0$. Put $\Gamma=\{\{q,p\}\}$, and Γ is nothing but $\mathrm{range}(J\tilde{S})$. Since Γ is not null by supposition, $\langle q,p\rangle$ does not vanish. Now, since $I(x)$ is a first integral, it must hold that

$$H\cdot SJ=SJ\cdot H. \tag{9}$$

Multiplying (9) by q and p from the right and using $\langle q,p\rangle\neq 0$, we can see that Hq and Hp are expressed as linear combinations of q and p. Then, we put

$$Hq=\alpha q+\beta p,\qquad Hp=\gamma q+\delta p, \tag{10}$$

where α, β, γ, and δ are constants. Next, multiplying (9) by ${}^t q$ and ${}^t p$ from the left and using (10) and $(q,p)=0$, we have $(\alpha{}^t q+\gamma{}^t p)J=-(\alpha{}^t q+\beta{}^t p)J$, and $(\beta{}^t q+\delta{}^t q)J=-(\gamma{}^t q+\delta{}^t p)J$. Then, it holds that $\alpha=\delta=0$ and $\beta+\gamma=0$, and accordingly, that $Hq=-\beta p$ and $Hp=\beta q$. Due to the supposition $H(\Gamma)\neq\{0\}$, β is not equal to zero. Thus, the conclusion is obtained from Proposition 1.

The next corollary follows directly from the above proof.

Corollary 4. *In* Γ *there exists a nonzero vector* q *which is orthogonal to* Hq *with respect to* (,).

Now, when Γ is null, the value of $I(x)$ on Γ remains zero. Otherwise, $I(x)$ gives a positive definite form when restricted to Γ, and its level curve on Γ is nothing but an integral curve of (1). Furthermore, the Hamiltonian vector field with the Hamiltonian $I(x)$ is given by $H_1x=SJx$. Then, it follows from (4) and (7.a) that H_1x is equal to $\langle p,q\rangle/\beta\cdot Hx$ for every $x\in\Gamma$. That is, as far as we are restricted to Γ, the symmetry generated by $I(x)$ produces the elliptic orbit of the original system (1). Moreover, $\ker(H_1)$ is equal to $J(\Gamma^{\perp})$, and this symmetry yields no action in any direction in $J(\Gamma^{\perp})$.

We close this paper by stressing again that an integral curve on Γ is nothing but a level curve of $I(x)$ if Γ is not null. If a discrete-time system which approximates (1) inherits the first integral given by (6) and leaves Γ invariant, the discrete orbit starting from a point in Γ, namely, a point sequence, lies on the solution curve of (1) itself. In addition, if the discrete system is sufficiently near the identity mapping, it results that at least in every Γ_β the phase portrait of (1) is reproduced accurately by the discrete-time system.

Reference

[1] V. I. Arnold: Mathematical Methods of Classical Mechanics (Engl. Trans. by K. Vogtmann and A. Weinstein). Springer (1978).