18. A Note on a Paper of Iwasawa

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1. Let F be a finite extension of a finite algebraic number field k and denote by C_k and C_F the ideal class group of k and of F respectively. A subgroup A of C_k is said to capitulate in F if A is contained in the kernel of natural homomorphism $C_k \to C_F$. The principal ideal theorem states that C_k always capitulates in Hilbert's class field K over k. However, for some k, C_k already capitulates in a proper subfield M of K: $k \subseteq M \subsetneqq K$. Such a field M exists if and only if there is a prime number p such that $C_{k,p}$ (=the p-class group of k) capitulates in a proper subfield F of Hilbert's p-class field K_p over k: $k \subseteq F \subsetneqq K_p$ (cf. [1]). In his paper ([1]), Iwasawa gave simple examples of k such that the 2-class group $C_{k,2}$ already capitulates in a proper subfield F of Hilbert's 2-class field K_2 over k.

Iwasawa's example. Let p, p_1, p_2 be 3 distinct prime numbers such that

i) $p \equiv p_1 \equiv p_2 \equiv 1$ mod. 4 and Legendre symbols

$$\left(\frac{p}{p_1}\right)=\left(\frac{p}{p_2}\right)=-1$$

ii) the norm of the fundamental unit of the real quadratic field $k'=\boldsymbol{Q}(\sqrt{p_1p_2})$ is 1.

Let

$$k=\boldsymbol{Q}(\sqrt{pp_1p_2}), \qquad K_2=\boldsymbol{Q}(\sqrt{p}, \sqrt{p_1}, \sqrt{p_2}),$$
$$F=\boldsymbol{Q}(\sqrt{p}, \sqrt{p_1p_2}).$$

Then K_2 is Hilbert's 2-class field over k and $C_{k,2}$ capitulates in the proper subfield F of K_2: $k \subseteq F \subsetneqq K_2$.

2. Let k and K_2 be as stated above. Then

$$F=\boldsymbol{Q}(\sqrt{p}, \sqrt{p_1p_2}), \quad F_1=\boldsymbol{Q}(\sqrt{p_1}, \sqrt{pp_2}), \quad F_2=\boldsymbol{Q}(\sqrt{p_2}, \sqrt{pp_1})$$

are all proper subfields of K_2 over k. In the following, we shall consider a question whether $C_{k,2}$ capitulates also in F_1 or in F_2.

Proposition 1. *Let $K_2^{(2)}$ denote Hilbert's 2-class field over K_2.*

i) *If $K_2=K_2^{(2)}$, $C_{k,2}$ capitulates also both in F_1 and in F_2.*

ii) *If $K_2 \neq K_2^{(2)}$, $C_{k,2}$ capitulates neither in F_1 nor in F_2.*

Proof. This is a consequence of Theorem 2 in [2].

Corollary. *$C_{k,2}$ capitulates in $F_1 \Longleftrightarrow C_{k,2}$ capitulates in F_2.*

Proposition 2. *Let $h_2(F)$ be the 2-class number of F. Then*

i) $K_2=K_2^{(2)} \Longleftrightarrow h_2(F)=2$.

ii) $K_2 \neq K_2^{(2)} \Longleftrightarrow 4 \mid h_2(F)$.

Proof. We shall prove ii) from which i) follows. Suppose $K_2 \neq K_2^{(2)}$. Since $Gal(K_2/k)$ is the four group, there is a (unique) subfield L of $K_2^{(2)}$ such that $K_2 \subseteq L \subseteq K_2^{(2)}$ and $Gal(L/k)$ is a nonabelian group of order 8 (cf. [2]). Then L/F is an unramified abelian extension of degree 4 whence $4 \mid h_2(F)$. The converse is obvious.

Remark. Replacing F by F_1 or F_2, we see that

i) $h_2(F_1)=2 \Longleftrightarrow K_2=K_2^{(2)} \Longleftrightarrow h_2(F_2)=2$,

ii) $4 \mid h_2(F_1) \Longleftrightarrow K_2 \neq K_2^{(2)} \Longleftrightarrow 4 \mid h_2(F_2)$

where $h_2(F_i)$ = the 2-class number of F_i $(i=1,2)$.

3. Let

K = a real bicyclic biquadratic extension of $\boldsymbol{Q}$,

E_K = the unit group of K,

k_i $(i=1,2,3)$ = the 3 quadratic subextensions of $K/\boldsymbol{Q}$,

ε_i = the fundamental unit of k_i $(i=1,2,3)$,

$h_2(K)$, $h_2(k_i)$ denote the 2-class numbers of K, k_i $(i=1,2,3)$.

Let

$Q(K)=[E_K : \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle]$

= the group index of $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ in E_K, where $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ is the subgroup of E_K generated by $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ± 1.

Then, it is known that $Q(K)=1$, 2 or 4 and

$$h_2(K)=\frac{1}{4}Q(K)h_2(k_1)h_2(k_2)h_2(k_3).$$

$(h(K)=\frac{1}{4}Q(K)h(k_1)h(k_2)h(k_3)$ for the class numbers) ([3]). Furthermore, a system of fundamental units of K is one of the following types ([3])

i) $\varepsilon_1, \varepsilon_2, \varepsilon_3$

ii) $\sqrt{\varepsilon_1}, \varepsilon_2, \varepsilon_3$ $(N\varepsilon_1=1)$

iii) $\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}, \varepsilon_3$ } $(N\varepsilon_1=N\varepsilon_2=1)$

iv) $\sqrt{\varepsilon_1\varepsilon_2}, \varepsilon_2, \varepsilon_3$ }

v) $\sqrt{\varepsilon_1\varepsilon_2}, \sqrt{\varepsilon_3}, \varepsilon_2$ }

vi) $\sqrt{\varepsilon_1\varepsilon_2}, \sqrt{\varepsilon_2\varepsilon_3}, \sqrt{\varepsilon_3\varepsilon_1}$ } $(N\varepsilon_1=N\varepsilon_2=N\varepsilon_3=1)$

vii) $\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}, \varepsilon_2, \varepsilon_3$ }

viii) $\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}, \varepsilon_2, \varepsilon_3$ $(N\varepsilon_1=N\varepsilon_2=N\varepsilon_3=-1)$

where $N\varepsilon_i$ is the abbreviation of $N_{k_i/\boldsymbol{Q}}(\varepsilon_i)$ $(i=1,2,3)$.

4. Let p, p_1, p_2 be 3 distinct prime numbers satisfying the conditions i), ii) in Iwasawa's example and let

$$k=\boldsymbol{Q}(\sqrt{pp_1p_2}), \qquad K_2=\boldsymbol{Q}(\sqrt{p}, \sqrt{p_1}, \sqrt{p_2}),$$
$$F=\boldsymbol{Q}(\sqrt{p}, \sqrt{p_1p_2})$$

as before.

Lemma 1. *Let $\varepsilon(k)$, $\varepsilon(p)$ and $\varepsilon(p_1p_2)$ denote fundamental units of k, $\boldsymbol{Q}(\sqrt{p})$ and $\boldsymbol{Q}(\sqrt{p_1p_2})$ respectively. Then, $\{\varepsilon(k), \varepsilon(p), \varepsilon(p_1p_2)\}$ is a fundamental system of units of F and $Q(F)=1$.*

Proof. We set $\varepsilon_1=\varepsilon(k)$, $\varepsilon_2=\varepsilon(p)$, $\varepsilon_3=\varepsilon(p_1p_2)$. Since $N\varepsilon_1=N\varepsilon_2=-1$, $N\varepsilon_3=1$, either $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ or $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$ is a fundamental system of units of F.

Suppose $\sqrt{\varepsilon_3} \in F$, then $F=\boldsymbol{Q}(\sqrt{p_1p_2})(\sqrt{\varepsilon_3})$. Hence, only prime divisors of $\boldsymbol{Q}(\sqrt{p_1p_2})$ lying above the rational prime 2 may ramify for $F/\boldsymbol{Q}(\sqrt{p_1p_2})$. But, this is a contradiction because prime divisors of $\boldsymbol{Q}(\sqrt{p_1p_2})$ lying above p ramify for $F/\boldsymbol{Q}(\sqrt{p_1p_2})$, so the result follows.

In the following, we shall denote by $h_2(d)$ the 2-class number of quadratic field $\boldsymbol{Q}(\sqrt{d})$.

Since $h_2(p)=1$ and $h_2(pp_1p_2)=4$, we have the following.

Corollary 1. $h_2(F)=h_2(p_1p_2)$ (≥ 2).

Corollary 2. i) $h_2(F)=2 \Longleftrightarrow h_2(p_1p_2)=2$

$$\Longleftrightarrow \left(\frac{p_1}{p_2}\right)_4\left(\frac{p_2}{p_1}\right)_4=-1$$

ii) $4\,|\,h_2(F) \Longleftrightarrow 4\,|\,h_2(p_1p_2) \Longleftrightarrow \left(\frac{p_1}{p_2}\right)_4=\left(\frac{p_2}{p_1}\right)_4=1$

where $(-)_4$ *denotes the biquadratic residue symbol.*

Proof. The 2nd equivalence follows from Proposition 3.3 in [5].

Remark. Let $F_1=\boldsymbol{Q}(\sqrt{p_1}, \sqrt{pp_2})$ and $F_2=\boldsymbol{Q}(\sqrt{p_2}, \sqrt{pp_1})$ as before, then

$$h_2(F_1)=\frac{1}{4}Q(F_1)h_2(pp_1p_2)h_2(p_1)h_2(pp_2)$$
$$=\frac{1}{4}Q(F_1)\cdot 4\cdot 1\cdot 2=2Q(F_1).$$

Similarly, $h_2(F_2)=2Q(F_2)$.

Now, we may give an answer to our question raised in section 2.

Theorem. *Let k, K_2, F, F_1 and F_2 be such fields as stated in Iwasawa's example.*

(1) $C_{k,2}$ (*=the* 2-*class group of* k) *capitulates in all* F, F_1 *and* F_2 ($\Longleftrightarrow C_{k,2}$ *capitulates in* $F_1 \Longleftrightarrow C_{k,2}$ *capitulates in* F_2)

$$\Longleftrightarrow h_2(F)=2 \Longleftrightarrow h_2(p_1p_2)=2 \Longleftrightarrow \left(\frac{p_1}{p_2}\right)_4\left(\frac{p_2}{p_1}\right)_4=-1$$
$$\Longleftrightarrow h_2(F_1)=2 \Longleftrightarrow Q(F_1)=1 \Longleftrightarrow h_2(F_2)=2 \Longleftrightarrow Q(F_2)=1.$$

(2) $C_{k,2}$ *capitulates in* F, *but neither in* F_1 *nor in* F_2

$$\Longleftrightarrow 4\,|\,h_2(F) \Longleftrightarrow 4\,|\,h_2(p_1p_2) \Longleftrightarrow \left(\frac{p_1}{p_2}\right)_4=\left(\frac{p_2}{p_1}\right)_4=1$$
$$\Longleftrightarrow 4\,|\,h_2(F_1) \Longleftrightarrow 2\,|\,Q(F_1) \Longleftrightarrow 4\,|\,h_2(F_2) \Longleftrightarrow 2\,|\,Q(F_2).$$

Corollary. *Let $\varepsilon(d)$ denote a fundamental unit of $\boldsymbol{Q}(\sqrt{d})$.*

i) $h_2(p_1p_2)=2 \Longleftrightarrow \{\varepsilon(pp_1p_2), \varepsilon(p_1), \varepsilon(pp_2)\}$ *is a fundamental system of units of* $F_1 \Longleftrightarrow \{\varepsilon(pp_1p_2), \varepsilon(p_2), \varepsilon(pp_1)\}$ *is a fundamental system of units of* F_2.

ii) $4\,|\,h_2(p_1p_2) \Longleftrightarrow \{\sqrt{\varepsilon(pp_1p_2)\varepsilon(p_1)\varepsilon(pp_2)}, \varepsilon(p_1), \varepsilon(pp_2)\}$ *is a fundamental system of units of* $F_1 \Longleftrightarrow Q(F_1)=2 \Longleftrightarrow h_2(F_1)=4 \Longleftrightarrow$ *similar equivalent conditions for* F_2.

Proof. Set $\varepsilon_1=\varepsilon(pp_1p_2)$, $\varepsilon_2=\varepsilon(p_1)$, $\varepsilon_3=\varepsilon(pp_2)$. Since $N\varepsilon_1=N\varepsilon_2=N\varepsilon_3=-1$, either $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ or $\{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}, \varepsilon_2, \varepsilon_3\}$ is a fundamental system of units of F_1, so the results follow for F_1 (similarly for F_2).

Example. Let $(p_1, p_2)=(13, 17)$ and let $p\equiv 1$ mod. 4 be any prime number satisfying $\left(\frac{p}{13}\right)=\left(\frac{p}{17}\right)=-1$ (for example, $p\equiv 5$ mod. $4\cdot 13\cdot 17$). The norm of the fundamental unit of $\boldsymbol{Q}(\sqrt{13\cdot 17})$ is 1 and its class number is 2. In this case

$$k=\boldsymbol{Q}(\sqrt{13\cdot 17\cdot p}), \qquad K_2=\boldsymbol{Q}(\sqrt{p}, \sqrt{13}, \sqrt{17})$$

$$F=\boldsymbol{Q}(\sqrt{p}, \sqrt{13\cdot 17}), \quad F_1=\boldsymbol{Q}(\sqrt{13}, \sqrt{17\cdot p}), \quad F_2=\boldsymbol{Q}(\sqrt{17}, \sqrt{13\cdot p})$$

and $C_{k,2}$ capitulates in all proper subfields F, F_1 and F_2 of K_2.

Let $(p_1, p_2)=(13, 53)$ and let $p\equiv 1$ mod. 4 be any prime number satisfying $\left(\frac{p}{13}\right)=\left(\frac{p}{53}\right)=-1$ (for instance, $p\equiv 5$ mod. $4\cdot 13\cdot 53$). The norm of the fundamental unit of $\boldsymbol{Q}(\sqrt{13\cdot 53})$ is 1 and its class number is 4. In this case

$$k=\boldsymbol{Q}(\sqrt{13\cdot 53\cdot p}), \qquad K_2=\boldsymbol{Q}(\sqrt{p}, \sqrt{13}, \sqrt{53}),$$

$$F=\boldsymbol{Q}(\sqrt{p}, \sqrt{13\cdot 53}), \quad F_1=\boldsymbol{Q}(\sqrt{13}, \sqrt{53\cdot p}), \quad F_2=\boldsymbol{Q}(\sqrt{53}, \sqrt{13\cdot p})$$

and $C_{k,2}$ capitulates in F, but neither in F_1 nor in F_2.

References

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