14. *Fractal Aspects of Localization of Algebraic Integers and Complex Dynamical Systems*

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(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 13, 1990)

1. Introduction. In this paper, we will consider the following problem.

Problem. *Let D be a compact set in $\boldsymbol{C}$. Characterize the properties of algebraic integers contained in D together with all their conjugates.*

Our problem is closely related to the Diophantine moment problem in the ferromagnetic Ising model and the theory of arithmetic holomorphic functions (see [6]).

An answer to our problem has been given by M. Fekete in 1923. He obtained the following theorem.

Theorem 1 (Fekete). *Suppose that D is a compact set in $\boldsymbol{C}$ with transfinite diameter less than* 1. *Then there exist only finitely many algebraic integers which are contained in D together with all their conjugates over the rational number field $\boldsymbol{Q}$.*

If the transfinite diameter is equal to 1, then there exist in general infinitely many such algebraic integers. A typical example is the following celebrated Kronecker's unit theorem (see [3]).

Theorem 2 (Kronecker). *Let D be the unit disc with center* 0 *in $\boldsymbol{C}$. If an algebraic integer is contained in D together with all its conjugates over $\boldsymbol{Q}$, then it is either* 0 *or one of the nth roots of* 1 *for some natural number n.*

For the details of the transfinite diameter, we refer the reader to [2]. For a general compact set D, Problem is still unsolved. But recently from the view point of the theory of complex dynamical systems, it is understood that the localization of algebraic integers is related to the filled-in Julia set of a monic polynomial and to the Mandelbrot set of the monic quadratic polynomials. In 1984, P. Moussa, J.S. Geronimo and D. Bessis obtained the following theorem (see [5]).

Definition 1. *Let T be a polynomial of degree at least* 2. *Then the filled-in Julia set K_T of T is the set of all complex numbers which do not escape to ∞ under the iterations of T.*

Definition 2. *We will denote the nth iterate of T by T^n. A complex number z is said to be a preperiodic point of T, if there exists integers $k>0$ and $l\geq 0$ such that $T^{k+l}(z)=T^l(z)$.*

Theorem 3 (M-G-B). *Let K_T be the filled-in Julia set of a monic polynomial T of degree at least* 2 *with rational integer coefficients. Then the set* *of all preperiodic points of T coincides with the set of all algebraic integers contained in K_T together with all their conjugates over $\boldsymbol{Q}$.*

It is known that the transfinite diameter of the filled-in Julia set of a monic polynomial is equal to 1. Hence Theorem 3 (M-G-B) is a natural generalization of Kronecker's theorem. In fact, we have only to take z^2 as $T(z)$ in Theorem 3 to obtain Kronecker's theorem.

In their papers referred as [5] and [6], they showed some results by generalizing Theorem 3 to polynomials with algebraic integer coefficients. But the conditions on their theorems are complicated. We obtain, by using the Galois theory, simpler theorems, which will be stated in section 2. Difficulties come from non-discreteness of the ring of algebraic integers in an extention field F and from the difference between an integral extention ring and an integrally closed integral domain.

2. Main results. 2.1. Discrete case.

Definition 3. *Let R be a ring in $\boldsymbol{C}$. If R has the following property (A), then we say that R has discreteness.*

(A) *If the absolute value $|x|$ of an element x of R is less than* 1, *then x is equal to* 0.

Lemma 1. *Let F be an algebraic number field of finite degree over $\boldsymbol{Q}$ and $\mathcal{O}_F$ denote the ring of algebraic integers in F. Then $\mathcal{O}_F$ has discreteness if and only if F is the rational numbers field or an imaginary quadratic field.*

In what follows, F denotes $\boldsymbol{Q}$ or an imaginary quadratic field. We obtained the following results.

Theorem 4. *Suppose that T is a monic polynomial of degree at least* 2 *with coefficients in $\mathcal{O}_F$ and K_T is the filled-in Julia set of T. Then the set of all preperiodic points of T coincides with the set of all algebraic integers contained in K_T together with all conjugates over F.*

Theorem 5. *Let T be a monic quadratic polynomial with coefficients in $\mathcal{O}_F$. Then K_T is connected if and only if T coincides with one of the polynomials, listed in Table in Appendix, up to inner automorphisms by elements of $PSL(2:\mathcal{O}_F)$.*

Remark 1. *If we take $\boldsymbol{Q}$ as F, then, as a corollary of Theorem* 4, *we have Theorem* 3 *(M-G-B).*

2.2. General case.

In this subsection, Ω denotes the algebraic closure of $\boldsymbol{Q}$, $\mathcal{O}_\Omega$ the set of all algebraic integers and $\mathcal{G}$ the Galois group $Gal(\Omega/\boldsymbol{Q})$.

For $T(x)=\sum_{i=0}^{d} a_i x^i$ of $\mathcal{O}_\Omega[x]$, and an element σ of $\mathcal{G}$, we define $(\sigma T)(x)$ as follows:

$$(\sigma T)(x)=\sum_{i=0}^{d} \sigma(a_i)x^i.$$

Theorem 6. *Let T be a monic polynomial with coefficients in $\mathcal{O}_\Omega$. Then the set $\{\alpha \in \mathcal{O}_\Omega;\ \sigma(\alpha) \in K_{\sigma T}, \forall \sigma \in \mathcal{G}\}$ coincides with the set of all preperiodic points of T.*

Remark 2. *If $T(x)$ belongs to $\boldsymbol{Z}[x]$, then $\sigma(T)(x)$ is equal to $T(x)$. Therefore, as a corollary of Theorem* 6, *we have Theorem* 3 *(M-G-B). By* *using the coset decomposition of the Galois group, Theorem* 4 *of the discrete case is also obtained from Theorem* 6.

Appendix. Here is Table of quadratic polynomials of $\mathcal{O}_{Q(\sqrt{m})}$.

Let ω be as follows;

$$\omega=\begin{cases}\dfrac{-1+\sqrt{m}}{2} & \text{if } m\equiv 1 \bmod 4\\ \sqrt{m} & \text{if } m\equiv 2 \bmod 4\\ \sqrt{m} & \text{if } m\equiv 3 \bmod 4\end{cases}$$

Then the set $\{1, \omega\}$ is an integral base of $\mathcal{O}_{Q(\sqrt{m})}$.

Table

$\boldsymbol{Q}$	$\Gamma=\{z^2, z^2-1, z^2-2, z^2+z, z^2+z+1, z^2+z-2\}$	
$\boldsymbol{Q}(\sqrt{-1})$	$\Gamma\cup\{z^2\pm\omega, z^2+\omega z, z^2+\omega z-\omega\}$ $\cup\{z^2+(1+\omega)z-1, z^2+(1+\omega)z-2\}$	
$\boldsymbol{Q}(\sqrt{-2})$	$\Gamma\cup\{z^2+(1+\omega)z-1, z^2+(1+\omega)z-2\}$	
$\boldsymbol{Q}(\sqrt{-3})$	$\Gamma\cup\{z^2+\omega z, z^2+\omega z-\omega, z^2+\omega z-(1+\omega)\}$ $\cup\{z^2+(1+\omega)z, z^2+(1+\omega)z-1, z^2+(1+\omega)z-(1+\omega)\}$	
$\boldsymbol{Q}(\sqrt{-5})$	$\Gamma\cup\{z^2+(1+\omega)z-2\}$	
$\boldsymbol{Q}(\sqrt{-6})$	$\Gamma\cup\{z^2+(1+\omega)z-2\}$	
$\boldsymbol{Q}(\sqrt{-7})$	$\Gamma\cup\{z^2+\omega z-1, z^2+(1+\omega)z-1\}$	
$\boldsymbol{Q}(\sqrt{-11})$	$\Gamma\cup\{z^2+\omega z-1, z^2+(1+\omega)z-1\}$	
$\boldsymbol{Q}(\sqrt{-15})$	$\Gamma\cup\{z^2+\omega z-1, z^2+(1+\omega)z-1\}$	
$\boldsymbol{Q}(\sqrt{-19})$	$\Gamma\cup\{z^2+\omega z-1, z^2+(1+\omega)z-1\}$	
$\boldsymbol{Q}(\sqrt{m})$	Γ	for other m

Remark. *If we substitute* $PSL(2:\mathcal{O}_F)$ *with* $PSL(2:\boldsymbol{C})$, *then the set of polynomials of* $\boldsymbol{Q}(\sqrt{-5})$ *and of* $\boldsymbol{Q}(\sqrt{-6})$ *in Table are contained in the set of polynomials of* $\boldsymbol{Q}(\sqrt{-1})$ *and of* $\boldsymbol{Q}(\sqrt{-3})$ *in Table, respectively.*

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