13. Spectral Analysis for the Casimir Operator on the Quantum Group $SU_q(1,1)$

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In this letter we will determine the spectrum of the Casimir operator for zonal spherical functions on the quantum group $SU_q(1,1)$, and show the Plancheral formula and the expansion theorem for them.

§1. $\mathcal{U}_q(su(1,1))$ $(0<q<1)$ be the real form of the universal quantum enveloping algebra [3] with the $*$-structure, $k^*=k$, $e^*=-f$, $f^*=-e$ [1]. In [1] we have classified irreducible unitary representations of $\mathcal{U}_q(su(1,1))$ as follows: We set a left $\mathcal{U}_q(su(1,1))$-module $V_l=\oplus_{i\in I_l}C\xi_i$ by

(1.1)
$$k\cdot\xi_j=q^{-j}\xi_j,$$
$$e\cdot\xi_j=q^{1/2-l}\frac{1-q^{2(l-j+1)}}{1-q^2}\xi_{j-1},$$
$$f\cdot\xi_j=q^{1/2-l}\frac{1-q^{2(l+j+1)}}{1-q^2}\xi_{j+1}.$$

Here the complex spin l and the indices set I_l are listed in the table below $(q=e^{-h},\ h>0)$.

(1.2)

l	I_l
$l=-1/2$	$I_l=\{1/2, 3/2, \cdots\}$, $\{-1/2, -3/2, \cdots\}$
$-1/2<l<0$	$I_l=Z$
$l\in 1/2N$	$I_l=\{l+1, l+2, \cdots\}$, $\{-l-1, -l-2, \cdots\}$
$l=-1/2+i\theta$	$(0\le\theta\le\pi/2h)$ $I_l=Z$, $(0<\theta\le\pi/2h)$ $I_l=Z+1/2$
$l=-1/2+i(\pi/2h+it)$ $(0<t)$	$I_l=Z$, $Z+1/2$

Let $w_{ij}^{(l)}$ be the matrix elements on $SU_q(1,1)$ corresponding to each representations. They are represented in terms of the basic hypergeometric functions

$${}_2\varphi_1\left(\begin{matrix}a,\ b\\ c\end{matrix};\ q,\ z\right)=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}z^n,$$

where $(a;q)_n=\prod_{r=0}^{n-1}(1-aq^r)$ $(1\le n\le\infty)$. For example, in the case of $i+j\le 0$, $j\le i$.

(1.3)
$$w_{ij}^{(l)}=q^{(j-i)(l+j)}\frac{(q^{2(l-i+1)};q^2)_{i-j}}{(q^2;q^2)_{i-j}}x^{-i-j}v^{i-j}{}_2\varphi_1\left(\begin{matrix}q^{2(l-j+1)},\ q^{-2(l+j)}\\ q^{2(i-j+1)}\end{matrix};\ q^2, q^2\zeta\right),$$

where x, u, v and y are the coordinate elements on $SU_q(1,1)$ and $\zeta=-q^{-1}uv$. These matrix elements satisfy the eigen-equation

(1.4) $$\pi_l(C)w_{ij}^{(l)}=[l+1/2]^2 w_{ij}^{(l)}$$

where π_l is the left invariant differential representation of $\mathcal{U}_q(sl(2))$ on $\mathcal{A}$ (the dual space of $\mathcal{U}_q(sl(2))$), and

(1.5) $$C=\frac{qk^2+q^{-1}k^{-2}-2}{(q-q^{-1})^2}+fe$$

is the Casimir element of $\mathcal{U}_q(sl(2))$ [3], and $[a]=(q^a-q^{-a})/(q-q^{-1})$.

Equation (1.4) is, in fact, a q-difference equation of the second order. In particular, the zonal spherical function

(1.6) $$w_{00}^{(l)}={}_2\varphi_1\left(\begin{matrix}q^{2l+2},\ q^{-2l}\\ q^2\end{matrix};\ q^2,\ q^2\zeta\right)$$

satisfies a q-analogue of the Legendre equation

(1.7) $$qD_{q^2}\{z(z+1)D_{q^2}T_{q^2}^{-1}\varphi(z)\}+[1/2]^2\varphi(z)=[l+1/2]^2\varphi(z)$$

where $z=-\zeta$, and

$$D_{q^2}\varphi(z)=\frac{\varphi(z)-\varphi(q^2z)}{(1-q^2)z},\qquad T_{q^2}\varphi(z)=\varphi(q^2z).$$

§ 2. We will consider the spectral theory for the difference equation which arises from the equation (1.7).

For a solution $\varphi(z)$ to (1.7), we set $\varphi(n)=\varphi(q^{2n})$. Then we see that $\varphi=(\varphi(n))_{n\in Z}$ solves the following difference equation

(2.1) $$(q+q^{1-2n})\varphi(n-1)-2(q^2+q^{1-2n})\varphi(n)+(q^3+q^{1-2n})\varphi(n+1)=\lambda\varphi(n),$$

where n runs over Z, and

(2.2) $$\lambda=(1-q^2)^2[l+1/2]^2.$$

Taking into account the Haar measure on $SU_q(1,1)$ (cf. [2]), we introduce a Hilbert space

$$l_{00}^2=\{\varphi=(\varphi(n))_{n\in Z}\mid \|\varphi\|_{00}<+\infty\}$$

with an inner product

(2.3) $$(\varphi,\ \psi)=(1-q^2)\sum_{n=-\infty}^{+\infty}\varphi(n)\overline{\psi(n)}q^{2n}.$$

It is easy to see that the equation (2.1) is formally self-adjoint in l_{00}^2. Moreover we impose on (2.1) the following boundary condition;

(2.4) $$\lim_{n\to+\infty}\{\varphi(n)-\varphi(n-1)\}=0.$$

Theorem 1. *Equation* (2.1) *with the boundary condition* (2.4) *constitutes a self-adjoint boundary value problem in* l_{00}^2.

Theorem 2. *The Green kernel* $G(n,m;\lambda)$ *for the above boundary value problem is given as follows*:

(2.5) $$G(n,m;\lambda)=\varphi_{+\infty}(n;\lambda)\cdot\varphi_{-\infty}(m;\lambda)\qquad(m\leq n)$$

where, setting $l=-1/2+i\theta$,

(2.6) $$\varphi_{+\infty}(n;\lambda)={}_2\varphi_1\left(\begin{matrix}q^{1+2i\theta},\ q^{1-2i\theta}\\ q^2\end{matrix};\ q^2,\ -q^{2n+2}\right)$$

and $\varphi_{-\infty}(n;\lambda)=\gamma(\theta)\phi(q^{2n};\theta)$ *with*

(2.7) $$\phi(z;\theta)=(-q^{1-2i\theta};q^2)_\infty(q^{4-4i\theta};q^4)_\infty(q^2z)^{-1/2+i\theta}$$

$$\times {}_2\varphi_1\left(\begin{matrix} q^{1-2i\theta}, q^{1-2i\theta} \\ q^{2-4i\theta} \end{matrix}; q^2, -z^{-1}\right)$$

$$(2.8) \quad \gamma(\theta)=\frac{-2(q^4;q^4)_\infty q^{-2i\theta}}{(1-q^2)(q^2;q^4)_\infty(-q^{1+2i\theta};q^2)_\infty(q^{1-2i\theta};q^2)_\infty(-q^{1-2i\theta};q^2)_\infty}.$$

Investigating the singularities of the Green function (cf. [4]), we can determine the spectrum and the spectral measure of this boundary value problem, and finally establish the eigen-function expansion theorem.

Theorem 3. *Let* $\varphi(\theta)=(\varphi_{+\infty}(n;\lambda))_{n\in Z}$ *and* $\theta_k=\pi/2h+(2k+1)i/2$. *Then, for any* $f\in l^2_{00}$, *we have*

$$(2.9) \qquad \|f\|^2_{00}=\int_0^{2\pi/h} d\theta c(\theta)|(f,\varphi(\theta))_{00}|^2+\sum_{k=0}^{\infty} c_k|(f,\varphi(\theta_k))_{00}|^2,$$

and

$$(2.10) \qquad f=\int_0^{\pi/2h} d\theta c(\theta)\varphi(\theta)\cdot(f,\varphi(\theta))_{00}+\sum_{k=0}^{+\infty} c_k\varphi(\theta_k)(f,\varphi(\theta_k))_{00},$$

where

$$(2.11) \qquad c(\theta)=\frac{4q^2(q^4;q^4)^2_\infty(q^{4i\theta};q^4)_\infty(q^{-4i\theta};q^4)_\infty}{\pi(1-q^2)(q^2;q^4)^2_\infty(q^{2+4i\theta};q^4)_\infty(q^{2-4i\theta};q^4)_\infty},$$

$$(2.12) \qquad c_k=[2k+1].$$

The formula (2.9) is viewed as the Plancheral formula for zonal spherical functions on the quantum group $SU_q(1,1)$, and, in the classical case, (2.10) is known as the Fok-Mehler formula.

Extending the theory developed here to the whole of functions on $SU_q(1,1)$, one can determine the spectrum of the Casimir operator and establish the Plancheral formula in the L^2-space on $SU_q(1,1)$ [5]. In particular, one sees that the principal series of unitary representations are parametrized as follows:

(2.13) $l=-1/2+i\theta$ $(0\leq\theta\leq\pi/2h)$, the principal continuous series;
$l\in\frac{1}{2}N$, the first discrete series;
$l=-1/2+i\theta_k$ $(k\in N)$, the second discrete series.

After this work was completed, the author learned the paper [6] of L. L. Vaksman and L. I. Korodskii. They also obtained the formula (2.9).

References

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