

12. Asymptotic Expansion of the Bergman Kernel for Strictly Pseudoconvex Complete Reinhardt Domains in C^2

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Introduction. The purpose of this note is to present an explicit representation of Fefferman's asymptotic expansion of the Bergman kernel on the diagonal for bounded strictly pseudoconvex complete Reinhardt domains in C^2 . As a consequence, we obtain a proof of the so-called Ramadanov conjecture for this class of domains.

Let $K=K(z)$ denote the Bergman kernel restricted to the diagonal of $\Omega \times \Omega$, where Ω is a bounded strictly pseudoconvex domain in C^n with C^∞ boundary. It was shown by C. Fefferman [2] that

$$K = \frac{\varphi}{\lambda^{n+1}} + \psi \log(-\lambda) \quad \text{with } \varphi, \psi \in C^\infty(\bar{\Omega}),$$

where λ is a C^∞ defining function of $\Omega = \{\lambda < 0\}$. On the other hand, it is only the ball amongst bounded strictly pseudoconvex domains on which K is explicitly known, and $\psi = 0$ in this case. We are thus interested in an explicit example of Ω for which $\psi \neq 0$. The so-called Ramadanov conjecture asks whether Ω is a ball if and only if $\psi = 0$ (cf. [4]). A local version of this conjecture was recently solved affirmatively by D. Burns and C. R. Graham in case $n=2$ (cf. [3]).

Assuming in particular that Ω is a complete Reinhardt domain in C^2 , we introduce a C^∞ function $p: (-\infty, 0] \rightarrow \mathbf{R}$, which is a hodograph transform describing the boundary of Ω . Then, an asymptotic expansion of K is obtained after a normalization, where the coefficients are expressed in terms of p and its derivatives; furthermore, we can explicitly determine $\varphi \bmod O(\lambda^3)$ and $\psi|_{\partial\Omega}$. By using the expression for $\psi|_{\partial\Omega}$, we obtain a proof of the Ramadanov conjecture mentioned above for this class of domains.

The same topic as in the present note was discussed earlier by D. Boichu and G. Coeuré in [1], where they presented an asymptotic expansion of the Bergman kernel in a non-explicit way and attempted to prove the Ramadanov conjecture for the same class of domains as ours. It seems to the present author that their proof of the Ramadanov conjecture is incomplete (Lemme 7 in [1] is incorrect); nevertheless their idea is very useful to us. Namely, we basically follow their method of analysis. The crucial difference is the choice of p , which, together with some other minor modifications, enables us to obtain an explicit expansion.

§ 1. Statement of the result. Let Ω be a strictly pseudoconvex domain

in \mathbf{C}^2 with C^∞ boundary. We assume that Ω is a *complete Reinhardt domain*, that is, $z=(z_1, z_2) \in \Omega$ whenever $|z_1| \leq |w_1|$ and $|z_2| \leq |w_2|$ with some $w=(w_1, w_2) \in \Omega$. Setting

$$f(x) = \sup \{ \log |z_2|; (e^x, z_2) \in \Omega \} \quad \text{for } x \in I = (-\infty, x_+),$$

where $x_+ = \sup \{ \log |z_1|; (z_1, z_2) \in \Omega \text{ for some } z_2 \in \mathbf{C} \}$, we have a local defining function λ of Ω in the form

$$\lambda(z) = \log |z_2| - f(\log |z_1|)$$

for $z \in \mathbf{C}^2$ with $z_1 z_2 \neq 0$ and $\log |z_1| \in I$. Recalling that the strict pseudoconvexity of Ω implies $f'' < 0$, we introduce a new dependent variable $p \in C^\infty(f'(I))$, a *hodograph transform*, defined by

$$p(f'(x)) = f''(x) \quad \text{for } x \in I.$$

(It turns out that Ω is a ball if and only if $p(v) = 2v - 2v^2$.) Our result is:

Theorem. *For the Bergman kernel on the diagonal of $\Omega \times \Omega$*

$$K(z) \equiv \frac{1}{(2\pi)^2 |z_1 z_2|^2} \frac{p}{4} \left\{ \frac{2}{\lambda^3} + \frac{p^{(2)}}{2} \frac{1}{\lambda^2} + \frac{(pp^{(3)})'}{3!} \frac{1}{\lambda} - \frac{(p^2 p^{(4)})''}{4!} \log |\lambda| \right\}$$

mod $C^0(\bar{\Omega} \cap \{z_1 z_2 \neq 0\})$. Here \equiv denotes that the difference between the left and right sides is an element of $C^0(\bar{\Omega} \cap \{z_1 z_2 \neq 0\})$, and p and its derivatives are evaluated at $f'(x)$ with $x = x(z) = \log |z_1|$.

Actually, the proof of Theorem above gives a full asymptotic expansion of $|z_1 z_2|^2 K(z)$ with respect to $\partial_i^{2-k}(1/\lambda)$ for $k=0, 1, 2, \dots$ (cf. (3) in Section 2 below).

As an immediate consequence of Theorem above, we have:

Corollary. *Assume furthermore that Ω is bounded. Then, $\psi = 0$ on $\partial\Omega$ if and only if $\Omega = \{c_1 |z_1|^2 + c_2 |z_2|^2 < 1\}$ with $c_1, c_2 > 0$.*

§ 2. Outline of the proof of theorem. Since monomials form a complete orthogonal system in the space of L^2 holomorphic functions in Ω , it follows that $(2\pi)^2 |z_1 z_2|^2 K(z) = L_{1,f}(x, \lambda)$, where

$$L_{t,f}(x, \lambda) = \sum_{\alpha \in N^2} 2\alpha_2 \exp(2\alpha_2 \lambda) / h_{t,f}(\alpha; x), \quad t \in [0, 1],$$

$$h_{t,f}(\alpha; x) = \int_{I-x} \exp \{ 2\alpha_2 (\mu(\alpha; x)\xi + R_f(t\xi; x)\xi^2) \} d\xi,$$

$$R_f(\xi; x) = \frac{1}{\xi^2} (f(x+\xi) - f(x) - f'(x)\xi) = \int_0^1 (1-\tau) f''(x+\tau\xi) d\tau$$

with $I-x = \{ \xi \in \mathbf{R}; \xi + x \in I \}$ and $\mu(\alpha; x) = \alpha_1 / \alpha_2 + f'(x)$. We consider $L_{t,f}(x, \lambda)$ in a neighborhood of $(x_0, 0)$ with $x_0 \in I$ fixed. We restrict the summation over N^2 to a smaller index set A , and replace $f \in C^\infty(I)$ by $\tilde{f} \in C^\infty(\mathbf{R})$ satisfying

$$(1) \quad \tilde{f}(x) = f(x) \quad \text{near } x = x_0, \quad \tilde{f}'' < 0 \quad \text{on } \mathbf{R}.$$

The resulting function $L_{t,\tilde{f},A}$ at $t=1$ is equivalent to $L_{1,f}$ mod C^∞ as for as A includes $\{ \alpha \in N^2; |\mu(\alpha; x_0)| < \varepsilon \}$ for some $\varepsilon > 0$. Furthermore, given $k_0, l_0 \in N_0$ with $k_0 \geq 2l_0 + 8$ and $N_0 = \{0\} \cup N$, there exists \tilde{f} as in (1) such that if $A \subset \{ \alpha \in N^2; \mu(\alpha; x_0) < \varepsilon' \}$ for some ε' (with $\varepsilon \leq \varepsilon' < \infty$) then

$$L_{1,\tilde{f},A} \equiv \sum_{k=0}^{k_0-1} \frac{1}{k!} \partial_i^k L_{t,\tilde{f},A}|_{t=0} \text{ mod } C^{l_0},$$

where $\partial_t = \partial/\partial t$. In the following, we write f in place of \tilde{f} .

We differentiate the expression of $L_{t,f,A}$ term-by-term and then under the sign of integration. Recalling $f'' = p(f')$, we see that $\partial_t^k L_{t,f,A}|_{t=0}$ is a linear combination of $L_{f,A}^{l,\eta}$ with $|\eta| - 2l = k$ whose coefficients are polynomials in p and its derivatives. Here

$$L_{f,A}^{l,\eta}(x, \lambda) = \sum_{\alpha \in A} \exp(2\alpha_2 \lambda) (2\alpha_2)^{1+l} H_\eta^l(\alpha; x),$$

$$H_\eta^l(\alpha; x) = h_{0,f}(\alpha; x)^{-1-m} \prod_{i=1}^m \int_{\mathbf{R}} \xi^{\eta_i} \exp \left\{ 2\alpha_2 \left(\mu(\alpha; x) \xi + \frac{f''(x)}{2} \xi^2 \right) \right\} d\xi$$

for $l \in N_0$ and $\eta = (\eta_1, \dots, \eta_m) \in N_0^m$. It turns out that

$$(2) \quad L_{f,A}^{l,\eta}(x, \lambda) = \frac{1}{2\pi} (-f''(x))^{1-|\eta|} \partial_x^{2+l-|\eta|} \partial_s^\eta \left\{ \int_{\mathbf{R}} \left(\sum_{\alpha \in A} A_1^{\alpha_1} A_2^{\alpha_2} \right) d\xi \right\} \Big|_{s=0},$$

where $A_1 = \exp 2\sqrt{-1}\xi$ and $A_2 = \exp \{ f''(x)\xi^2 + 2\sqrt{-1}(f'(x) + f''(x)S_1(s))\xi + 2(\lambda - f''(x)S_2(s)) \}$ with

$$S_1(s) = \sum_j s_j, \quad S_2(s) = \sum_{i \leq j} s_i s_j \quad \text{for } s \in \mathbf{R}^m.$$

Evaluating the integral on the right side of (2) via residue calculus, we have:

Lemma. For $l \in N_0$ and $\eta \in N_0^m$,

$$L_{f,A}^{l,\eta}(x, \lambda) \equiv \frac{f''(x)}{4} (\partial_s^\eta \exp S_2(s)|_{s=0}) (-f''(x))^{-|\eta|/2} \partial_x^{2+l-|\eta|/2} \frac{1}{\lambda}$$

mod C^∞ , where ∂_x^{-1} denotes indefinite integration with respect to λ . In particular, if $|\eta|$ is odd, then $L_{f,A}^{l,\eta} \in C^\infty$.

Examining the coefficients of $L_{f,A}^{l,\eta}$ in the expression of $\partial_t^k L_{t,f,A}|_{t=0}$, we arrive at the asymptotic expansion mentioned in Section 1:

$$(3) \quad L_{1,f} \sim \frac{p}{4} \sum_{k=0}^{\infty} \frac{P_k}{p^k} \partial_\lambda^{2-k} \frac{1}{\lambda},$$

where P_k is a homogeneous polynomial of degree $2k$ and weight $2k$ in $p, p', \dots, p^{(2k)}$. Here the weight of the monomial $p^{(\beta_1)} \dots p^{(\beta_l)}$ is defined by $\beta_1 + \dots + \beta_l$. Our theorem will be obtained by evaluating explicitly P_k for $k = 0, 1, 2, 3$. The computations are lengthy, but are simplified by examining the Bergman kernel of a particular domain Ω , for example, $\Omega = \{z \in \mathbf{C}^2; |z_1|^2 |z_2|^{2q} + |z_2|^{2r} < 1\}$ with $q \in N_0$ and $r \in (0, \infty)$, for which $\psi = 0$ and $p(v) = 2v(1 + qv)\{1 + (q-r)v\}$.

Details will appear elsewhere.

References

- [1] D. Boichu and G. Coeuré: Sur le noyau de Bergman des domaines de Reinhardt. *Invent. Math.*, **72**, 131-152 (1983).
- [2] C. Fefferman: The Bergman kernel and biholomorphic mappings of pseudoconvex domains. *ibid.*, **26**, 1-65 (1974).
- [3] C. R. Graham: Scalar boundary invariants and the Bergman kernel. *Lect. Notes in Math.*, Springer, **1276**, 108-135 (1987).
- [4] I. P. Ramadanov: A characterization of the balls in C^n by means of the Bergman kernel. *C. R. Acad. Bulgare Sci.*, **34**, 927-929 (1981).