11. *On Pathwise Projective Invariance of Brownian Motion. III*

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In part I we demonstrated that the group $SL(2, \boldsymbol{R})$ acts on the path space of Brownian motion $\{B(t)\,;\, t \in \boldsymbol{R}\}$ as

$$B^g(t\,;\,\omega)=(ct+d)B\Big(\frac{at+b}{ct+d}\,;\,\omega\Big)-ctB\Big(\frac{a}{c}\,;\,\omega\Big)-dB\Big(\frac{b}{d}\,;\,\omega\Big),\qquad g=\begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \in SL(2, \boldsymbol{R}), \tag{1}$$

and that the above action is compatible with the group action,

$$(B^g)^h(t\,;\,\omega)=B^{gh}(t\,;\,\omega).$$

In this part we obtain analogous invariance properties of multi-parameter Brownian motions.

§8. Multi-parameter Brownian motion. A real Gaussian system $\{B(\boldsymbol{t})\,;\, \boldsymbol{t} \in \boldsymbol{R}^n\}$ is called a Brownian motion if it satisfies the following conditions ([3]):

($\mathscr{B}$1) $\qquad B(\boldsymbol{0})=0,$

($\mathscr{B}$2) $\qquad B(\boldsymbol{s})-B(\boldsymbol{t})\simeq N(0, |\boldsymbol{s}-\boldsymbol{t}|)$

and

($\mathscr{B}$3) $\qquad B(\boldsymbol{t})$ is continuous in $\boldsymbol{t}$.

It is easy to see that the following transformed processes $B_{1,\boldsymbol{v}}$, $B_{2,u}$ and $B_{4,g}$ satisfy the conditions ($\mathscr{B}$1)–($\mathscr{B}$3). That is, all these processes are Brownian motions.

($\mathscr{T}$1) $\qquad B_{1,\boldsymbol{v}}(\boldsymbol{t})\equiv B(\boldsymbol{t}+\boldsymbol{v})-B(\boldsymbol{v}), \qquad \boldsymbol{v} \in \boldsymbol{R}^n$ (shift),

($\mathscr{T}$2) $\qquad B_{2,u}(\boldsymbol{t})\equiv e^{-u/2}B(e^u\boldsymbol{t}), \qquad u \in \boldsymbol{R}^1$ (homogeneous dilation)

and

($\mathscr{T}$4) $\qquad B_{4,g}(\boldsymbol{t})\equiv B(g\cdot\boldsymbol{t}), \qquad g \in SO(n)$, (rotation).

A difficulty occurs when we consider about multi-parameter analogue of the transform ($\mathscr{T}$3). Let us recall that in the case of 1-parameter Brownian motion, we employed the transformation

($\mathscr{T}$3) $\qquad B_3(t)=t\cdot B\Big(-\dfrac{1}{t}\Big), \qquad t \in \boldsymbol{R}^1$

as the projective inversion and obtained the pathwise projective invariance. We can consider this inversion map as a coordinate map between 0-neighborhood $\{(x, y) \in \boldsymbol{P}^1, y\neq 0\}$ and ∞-neighborhood $\{(x, y)\,;\, x\neq 0\}$ of the 1-dimensional real projective space $\boldsymbol{P}^1$,

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$$(9_1)\qquad (x,1)\longrightarrow\left(1,\frac{1}{x}\right).$$

Let us proceed to the 2-dimensional case. One of the coordinate maps is

$$(9_2)\qquad (x,y,1)\longrightarrow\left(1,\frac{y}{x},\frac{1}{x}\right).$$

The corresponding process may be

$$(10)\qquad \tilde{B}(x,y)=\frac{|x|^{1/2}}{(1+y^2)^{1/4}}(x^2+y^2)^{1/4}B\left(\frac{1}{x},\frac{y}{x}\right).$$

Unfortunately $\{\tilde{B}(x,y);\,(x,y)\in\boldsymbol{R}^2\}$ is not a Brownian motion in the above sense (for example, calculate the variance of $\tilde{B}(1,1)-\tilde{B}((1/2),1)$).

The above fact suggests us that we cannot consider the parameter space of multi-parameter Brownian motion as a projective space and that we cannot expect the full invariance as Theorem 1. In fact the corresponding geometry is the Möbius geometry. The Brownian motion which corresponds to the projective inversion of Möbius geometry is

$$(\mathcal{T}3')\qquad B_3(\boldsymbol{t})\equiv|\boldsymbol{t}|B\left(\frac{\boldsymbol{t}}{|\boldsymbol{t}|^2}\right).$$

Thus, in this paper, we employ the transform $(\mathcal{T}3')$ above as the projective inversion.

§ 9. **Möbius sphere and conformal geometry.** Let $\underline{t}=\begin{pmatrix} y\\ \boldsymbol{t}\\ z\end{pmatrix}$, $\boldsymbol{t}\in\boldsymbol{R}^n$, $y,\ z\in\boldsymbol{R}$, expresses the point $\underline{t}$ of $(n+1)$-dimensional real projective space $\boldsymbol{P}^{n+1}(\boldsymbol{R})$ in the homogeneous coordinate. Let us introduce an indefinite inner product $\langle\underline{t},\underline{s}\rangle={}^t\underline{t}J\underline{s}$ in $\boldsymbol{P}^{n+1}$, where $J=\left(\begin{array}{c|c|c} 0 & & -1\\ \hline & \boldsymbol{I}_n & \\ \hline -1 & & 0\end{array}\right)$.

Note that, if we introduce the coordinate $\xi=\frac{1}{\sqrt{2}}(y+z)$ and $\eta=\frac{1}{\sqrt{2}}(y-z)$, the above inner product is identified with the indefinite inner product of type $\boldsymbol{I}_{n+1,1}$.

The subset $S^n\equiv\{\underline{t}\in\boldsymbol{P}^{n+1};\ \langle\underline{t},\underline{t}\rangle=0\}$ is called the Möbius sphere of radius 0. The group $\text{Mö}(n)=\{g\in GL(n+2,\boldsymbol{R});\ {}^tgJg=\alpha_gJ,\ \alpha_g\in\boldsymbol{R}\}$ makes S^n invariant and is called the n-dimensional Möbius group.

The neighborhood $\{z\neq0\}$ of S^n is identified with $\boldsymbol{R}^n$ as

$$(11_0)\qquad \begin{pmatrix}\frac{1}{2}|\boldsymbol{t}|^2\\ \boldsymbol{t}\\ 1\end{pmatrix}\in S^n\longleftrightarrow\boldsymbol{t}\in\boldsymbol{R}^n.$$

The local coordinate around the infinity point is similarly defined as

$$(11_\infty)\qquad \begin{pmatrix}1\\ \boldsymbol{t}\\ \frac{1}{2}|\boldsymbol{t}|^2\end{pmatrix}\simeq\boldsymbol{t}.$$

The actions of Mö(n) on the coordinate (11) are:

I) Rotation.

$$\left(\begin{array}{c|c|c} 1 & & 0 \\ \hline & g & \\ \hline 0 & & 1 \end{array}\right); \quad g\in SO(n): \quad \boldsymbol{t}\longrightarrow g\cdot\boldsymbol{t}, \tag{12}$$

II) Shift.

$$\left(\begin{array}{c|c|c} 1 & {}^t\boldsymbol{v} & \frac{1}{2}|\boldsymbol{v}|^2 \\ \hline 0 & \boldsymbol{I}_n & \boldsymbol{v} \\ \hline 0 & 0 & 1 \end{array}\right)\cdot\begin{pmatrix} \frac{1}{2}|\boldsymbol{t}|^2 \\ \boldsymbol{t} \\ 1 \end{pmatrix}=\begin{pmatrix} \frac{1}{2}|\boldsymbol{t}|^2+(\boldsymbol{v},\boldsymbol{t})+\frac{1}{2}|\boldsymbol{v}|^2 \\ \boldsymbol{t}+\boldsymbol{v} \\ 1 \end{pmatrix}, \qquad \boldsymbol{v}\in\boldsymbol{R}^n. \tag{13}$$

III) Homogeneous dilation.

$$\left(\begin{array}{c|c|c} e^u & & \\ \hline & e^{u/2}\boldsymbol{I}_n & \\ \hline & & 1 \end{array}\right): \quad \boldsymbol{t}\longrightarrow e^{u/2}\boldsymbol{t}, \quad u\in\boldsymbol{R}. \tag{14}$$

IV) Shift around the ∞. Let us consider the action of the shift operator in the coordinate (11_∞).

$$\left(\begin{array}{c|c|c} 1 & {}^t\boldsymbol{v} & \frac{1}{2}|\boldsymbol{v}|^2 \\ \hline 0 & \boldsymbol{I}_n & \boldsymbol{v} \\ \hline 0 & 0 & 1 \end{array}\right)\cdot\begin{pmatrix} 1 \\ \boldsymbol{t} \\ \frac{1}{2}|\boldsymbol{t}|^2 \end{pmatrix}=\begin{pmatrix} 1+(\boldsymbol{v},\boldsymbol{t})+\frac{1}{4}|\boldsymbol{v}|^2|\boldsymbol{t}|^2 \\ \boldsymbol{t}+\frac{1}{2}|\boldsymbol{t}|^2\boldsymbol{v} \\ \frac{1}{2}|\boldsymbol{t}|^2 \end{pmatrix}\simeq\begin{pmatrix} 1 \\ \dfrac{\boldsymbol{t}+\frac{1}{2}|\boldsymbol{t}|^2\boldsymbol{v}}{1+(\boldsymbol{v},\boldsymbol{t})+\frac{1}{4}|\boldsymbol{v}|^2|\boldsymbol{t}|^2} \\ * \end{pmatrix}. \tag{13_∞}$$

Using the matrix $A=\begin{pmatrix} 0, & 0, & 1 \\ 0, & \boldsymbol{I}_n, & 0 \\ 1, & 0, & 0 \end{pmatrix}$ corresponds to the coordinate map between the 0-neighborhood and the ∞-neighborhood, we obtain the matrix form of above transform with respect to the 0-neighborhood as

$$A\left(\begin{array}{c|c|c} 1 & {}^t\boldsymbol{v} & \frac{1}{2}|\boldsymbol{v}|^2 \\ \hline 0 & \boldsymbol{I}_n & \boldsymbol{v} \\ \hline 0 & 0 & 1 \end{array}\right)A=\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline \boldsymbol{v} & \boldsymbol{I}_n & 0 \\ \hline \frac{1}{2}|\boldsymbol{v}|^2 & {}^t\boldsymbol{v} & 1 \end{array}\right). \tag{15}$$

The transformations I)–IV) gives us the explicit forms of the actions of the Mö(n) in this coordinate.

Note that in the case $d=1$ the action IV) is

$$t\longrightarrow\frac{t+\dfrac{1}{2}t^2v}{\dfrac{1}{4}t^2v^2+tv+1}=\frac{t}{\dfrac{1}{2}vt+1}. \tag{16}$$

This transformation is considered in [2] and is named as special conformal transformation.

§ 10. Projective invariance—Multi-parameter case.

10–1. Let us introduce two subgroups G_0 and G_∞ of Mö(n):

$$
(17) \qquad G_0 \equiv \left\{ \left(\begin{array}{c|c|c} e^{-u(n+2)/(n+1)} & e^{u(n+2)/(n+1)} \cdot {}^t\boldsymbol{v} & e^{u(n+2)/(n+1)} \cdot (|\boldsymbol{v}|^2/2) \\ \hline \mathbf{0} & e^{u/(n+1)} \cdot g & e^{u/(n+1)} \cdot g\boldsymbol{v} \\ \hline 0 & {}^t\mathbf{0} & e^{un/(n+1)} \end{array} \right); \quad \begin{array}{l} u \in \boldsymbol{R}, \\ g \in SO(n), \\ \boldsymbol{v} \in \boldsymbol{R}^n \end{array} \right\},
$$

and

$$
(18) \qquad G_\infty \equiv \{\boldsymbol{g} \in \text{Mö}(n);\ {}^t\boldsymbol{g} \in G_0\}.
$$

Then, we have the following theorem which describes the pathwise projective invariance of multi-parameter Brownian motion.

Theorem 7. *Let $\{B(\boldsymbol{t});\ \boldsymbol{t} \in \boldsymbol{R}^n\}$ be a Brownian motion then*

I–a) *for any $\boldsymbol{g} \in G_0$*

$$
B^{\boldsymbol{g}}(\boldsymbol{t}) \equiv e^{-u/2}\{B(e^u g\boldsymbol{t} + e^u g\boldsymbol{v}) - B(e^u g\boldsymbol{v})\}
$$

is again a Brownian motion, here we employ the parametrization of G_0 described in (17).

I–b) *The above action satisfies the composition law of the group*

$$
(B^{\boldsymbol{g}})^{\boldsymbol{h}}(\boldsymbol{t}) = B^{\boldsymbol{g}\cdot\boldsymbol{h}}(\boldsymbol{t}), \qquad \text{for any } \boldsymbol{g}, \boldsymbol{h} \in G_0.
$$

II–a) *For any $\boldsymbol{g} \in G_\infty$*

$$
B^{\boldsymbol{g}}(\boldsymbol{t}) \equiv e^{u/2}\left(1 + \langle \boldsymbol{v}, \boldsymbol{t}\rangle + \frac{1}{4}|\boldsymbol{t}|^2|\boldsymbol{v}|^2\right)^{1/2} \cdot B\left(\frac{e^{-u} g\left(\boldsymbol{t} + \frac{1}{2}|\boldsymbol{t}|^2 \boldsymbol{v}\right)}{1 + \langle \boldsymbol{v}, \boldsymbol{t}\rangle + \frac{1}{4}|\boldsymbol{t}|^2|\boldsymbol{v}|^2} \right) - \frac{1}{2} e^{u/2} |\boldsymbol{t}|\,|\boldsymbol{v}| \cdot B\left(\frac{2e^{-u} g\boldsymbol{v}}{|\boldsymbol{v}|^2} \right),
$$

here we use the similar parametrization as of G_∞ (17).

II–b) *The action satisfies the composition law*

$$
(B^{\boldsymbol{g}})^{\boldsymbol{h}}(\boldsymbol{t}) = B^{\boldsymbol{g}\cdot\boldsymbol{h}}(\boldsymbol{t}), \qquad \text{for any } \boldsymbol{g}, \boldsymbol{h} \in G_\infty.
$$

Note that the above actions of G_0 and G_∞ are not compatible as the group action of Mö(n). That is, there exist $\boldsymbol{g}, \boldsymbol{g}' \in G_0$ and $\boldsymbol{h}, \boldsymbol{h}' \in G_\infty$ such that they satisfy $\boldsymbol{gh} = \boldsymbol{h}'\boldsymbol{g}'$ as elements Mö(n) but the actions on the path space of the Brownian motion is not compatible, $(B^{\boldsymbol{g}})^{\boldsymbol{h}} \neq (B^{\boldsymbol{h}'})^{\boldsymbol{g}'}$.

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