84. On the Hasse Norm Principle for Certain Generalized Dihedral Extensions over Q

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Introduction. Let l be an odd prime number and put $l^* := (-1)^{(l-1)/2} l$. Set $k := Q(\sqrt{l^*})$ and let K be the Hilbert class field of k. In this note, we study the Hasse norm principle for the Galois extension K/Q whose Galois group is a generalized dihedral group. More precisely, we express the number knot group for K/Q in terms of the ideal class group of k. Our theorem says that the validity of the Hasse norm principle for K/Q is equivalent to that for K/k. As an application, we determine the Ono invariant $E(K/Q)$ ([3, 4]), which was the motivation of this work.

§1. The number knot group $Ш(K/Q)$. For a finite Galois extension L/F of number fields, we denote by $Ш(L/F)$ the number knot group $F^\times \cap NJ_L/NL^\times$, where J_L is the idele group of L and N means the norm map in the obvious sense. Clearly, $Ш(L/F)=\{0\}$ is equivalent to the fact that the Hasse norm principle holds for L/F and we also remark that $Ш(L/F)$ is nothing but the Tate-Shafarevich group of the norm torus $T := \operatorname{Ker}(R_{L/F}(G_m) \xrightarrow{N} G_m)$.

First, let us recall Tate's cohomological method ([6]) to study $Ш(L/F)$ for a finite Galois extension L/F of number fields with the Galois group $G := Gal(L/F)$. (See, for example [5].)

By the exact sequence of G-modules

$$0 \longrightarrow L^\times \longrightarrow J_L \longrightarrow C_L \longrightarrow 0 \tag{1.1}$$

where $C_L := J_L/L^\times$, we have an exact sequence of Tate cohomology groups

$$\cdots \longrightarrow \hat{H}^{-1}(G, J_L) \xrightarrow{f} \hat{H}^{-1}(G, C_L) \longrightarrow \hat{H}^0(G, L^\times) \xrightarrow{g} \hat{H}^0(G, J_L) \longrightarrow \cdots. \tag{1.2}$$

Here it is easy to see that

$$\operatorname{Coker} f \simeq \operatorname{Ker} g = Ш(L/F). \tag{1.3}$$

If we choose a place w of L lying over each place v of F and denote by G_w the decomposition group of w, then we have the following commutative diagram:

$$\begin{array}{ccc} \prod_v H_2(G_w, Z) & \xrightarrow{\phi} & H_2(G, Z) \\ \wr\downarrow & & \wr\downarrow \\ \prod_v \hat{H}^{-3}(G_w, Z) & \xrightarrow{\psi} & \hat{H}^{-3}(G, Z) \\ \wr\downarrow \prod_v \cup\alpha_w & & \wr\downarrow \cup\alpha \\ \hat{H}^{-1}(G, J_L) = \prod_v \hat{H}^{-1}(G_w, L_w^\times) & \xrightarrow{f} & \hat{H}^{-1}(G, C_L) \end{array} \tag{1.4}$$

where ϕ and ψ are the sum of the corestrictions with respect to $G_w \xrightarrow{\subset} G$

for each v and the lower vertical arrows are isomorphisms induced by the cup products with the canonical classes $\alpha \in H^2(G, C_L)$ and $\alpha_w \in H^2(G_w, L_w^\times)$ for each v.

By (1.3) and (1.4), the determination of Ш(L/F) is reduced to a purely group-theoretical problem:

(1.5) $$Ш(L/F) \simeq \mathrm{Coker}\,(\prod_v H_2(G_w, Z) \xrightarrow{\phi} H_2(G, Z)).$$

Now, let us come back to our case in the introduction. The Artin reciprocity map $\alpha_{K/k}$ identifies the ideal class group H_k of k and the Galois group $H := Gal(K/k)$. The genus theory tells us that the order of H_k, the class number h_k of k, is odd. So, if we choose τ of order 2 in G, then we have a semi-direct product:

$$G = H \cdot \langle \tau \rangle \qquad \text{with } H \text{ normal.}$$

Lemma 1. $\tau\sigma\tau^{-1} = \sigma^{-1}$ *for each* $\sigma \in H$

Proof. Since $\mathcal{P}\mathcal{P}^\tau = N_{k/Q}\mathcal{P}$ is principal for any prime ideal $\mathcal{P}$ of k, we have

$$\tau\alpha_{K/k}(\mathcal{P})\tau^{-1} = \alpha_{K/k}(\mathcal{P}^\tau) = \alpha_{K/k}(\mathcal{P})^{-1}.$$

Therefore, our claim follows from the Čebotarev density theorem for K/k.

Lemma 2. *Each* G_w *is cyclic and so* $H_2(G_w, Z) = 0$.

Proof. It is enough to show that G_w is cyclic for any w lying above l. Let $\mathfrak{l}$ be the restriction of w to k. Since $l = \mathfrak{l}^2$ and h_k is odd, $\mathfrak{l}$ is trivial in H_k and so $\alpha_{K/k}(\mathfrak{l}) = 1$. Hence G_w is cyclic of order 2.

By (1.5) and Lemma 2, we have

(1.6) $$Ш(K/Q) = H_2(G, Z) \quad \text{and} \quad Ш(K/k) = H_2(H, Z).$$

Proposition. $H_2(G, Z) = H_2(H, Z)$.

Proof. Consider the Lyndon-Hochschild-Serre spectral sequence

$$E^2_{p,q} = H_p(G/H, H_q(H, Z)) \Longrightarrow H_{p+q}(G, Z)$$

associated to the extension

(1.7) $$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1.$$

Let

$$H_n(G, Z) = F_{n,0} \supseteq F_{n,1} \supseteq \cdots \supseteq F_{n,m} \supseteq \cdots$$

be the filtration of $H_n(G, Z)$ such that $E^\infty_{n-p,p}$ is the p-th composition factor in $H_n(G, Z)$ and let

$$d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$$

be the differentials.

Since G/H is cyclic, $E^\infty_{2,0} \subset E^2_{2,0} = 0$ and so we have

(1.8) $$H_2(G, Z) = F_{1,1}.$$

In the exact sequence

(1.9) $$0 \longrightarrow E^\infty_{0,2} \longrightarrow F_{1,1} \longrightarrow E^\infty_{1,1} \longrightarrow 0$$

it is easy to see that

(1.10) $$E^\infty_{1,1} \simeq E^2_{1,1}/\mathrm{Im}\, d^2_{3,0}$$
$$E^\infty_{0,2} \simeq E^3_{0,2}/\mathrm{Im}\, d^3_{3,0}, \qquad E^3_{0,2} \simeq E^2_{0,2}/\mathrm{Im}\, d^2_{2,1}.$$

Here, remark that

(1.11) $$E^\infty_{3,0} = E^2_{3,0}$$

since the natural map $F_{3,0}=H_3(G,Z)\to H_3(G/H,Z)=E^2_{3,0}$ is surjective by the splitting of (1.7). Hence we have

(1.12) $$d^2_{3,0}=d^3_{3,0}=0.$$

Together with (1.8)–(1.12), we have an exact sequence

(1.13) $$E^2_{2,1}\xrightarrow{d^2_{2,1}}E^2_{0,2}\longrightarrow H_2(G,Z)\longrightarrow E^2_{1,1}\longrightarrow 0.$$

Here, since G/H is cyclic and h_k is odd, using Lemma 1, we can see that

(1.14) $$E^2_{1,1}=E^2_{2,1}=0 \quad\text{and}\quad E^2_{0,2}=H_2(H,Z).$$

Therefore, our claim follows from (1.13).

Let $H=\oplus_{i=1}^n C_i$ be the decomposition of H into the direct sum of the cyclic groups C_i $(1\leq i\leq n)$. Then, we can see

(1.15) $$H_2(H,Z)\simeq \wedge^2_Z H\simeq \oplus_{1\leq i<j\leq n}C_i\otimes_Z C_j$$

since the Tor term in the Künneth formula vanishes (cf. [5] Lemma 5, or [1] V. 6). Hence, by (1.6), Proposition and (1.15), we obtain

Theorem 1. *Keeping the above notation, the number knot group for $K/\boldsymbol{Q}$ is given by*

$$Ш(K/\boldsymbol{Q})\simeq Ш(K/k)\simeq \wedge^2_Z H_k\simeq \oplus_{1\leq i<j\leq n}C_i\otimes_Z C_j$$

Corollary 1. *The followings are equivalent:*

a) *The Hasse norm principle holds for $K/\boldsymbol{Q}$.*

b) *The Hasse norm principle holds for K/k.*

c) *The ideal class group of k is cyclic.*

Example. According to the table of Wada [7],

$$H_k\simeq Z/3Z\times Z/3Z,$$

for $l=4027$. Hence, we have

$$Ш(K/\boldsymbol{Q})\simeq Z/3Z.$$

§2. The Ono invariant $E(K/\boldsymbol{Q})$. Let L/F be a finite Galois extension of number fields. Ono [3, 4] introduced a kind of Euler number $E(L/F)$ defined by

$$E(L/F):=\frac{h_L}{h_F\cdot h_T}$$

where h_L, h_F and h_T are the class numbers of L, F and the norm torus $T:=\operatorname{Ker}(R_{L/F}(G_m)\xrightarrow{N}G_m)$. And he obtained an elementary cohomological expression for $E(L/F)$ which involves the number knot group for L/F.

In the following, we determine $E(K/\boldsymbol{Q})$ for our $K/\boldsymbol{Q}$ as an application of §1.

By Theorem in [3] §2, we have

(2.1) $$E(K/\boldsymbol{Q})=\frac{\# Ш(K/\boldsymbol{Q})\prod_v \#\hat{H}^0(G_w,\mathcal{O}_w^\times)}{[K':\boldsymbol{Q}](Z^\times:N_{K/\boldsymbol{Q}}\mathcal{O}_K^\times)}$$

where $\#:=$ the cardinality, $\mathcal{O}_K$ denotes the ring of integers in K, $\mathcal{O}_w$ denotes the ring of integers in K_w ($\mathcal{O}_w:=K_w$ for archimedean w) and K' is the maximal abelian subextension of $K/\boldsymbol{Q}$.

First, an easy calculation shows that the commutator subgroup of $G=H$ and so $[K':\boldsymbol{Q}]=2$.

Case 1. k is imaginary.

Clearly, $(\boldsymbol{Z}^{\times}: N_{K/\boldsymbol{Q}}\mathcal{O}_K^{\times})=2$. By local class field theory, $\prod_v \sharp \hat{H}^0(G_w, \mathcal{O}_w^{\times})=2^2$. Hence, by (2.1), we have

$$E(K/\boldsymbol{Q})=\sharp Ш(K/\boldsymbol{Q}).$$

Case 2. k is real.

Since H has an odd order, K is real and so $\prod_v \sharp \hat{H}^0(G_w, \mathcal{O}_w^{\times})=2$. By the genus theory, $N_{k/\boldsymbol{Q}}(\varepsilon)=-1$ for the fundamental unit ε of k. Therefore $N_{K/\boldsymbol{Q}}(\varepsilon)=(-1)^{h_k}=-1$ and so $(\boldsymbol{Z}^{\times}: N_{K/\boldsymbol{Q}}\mathcal{O}_K^{\times})=1$. Hence, by (2.1), we have

$$E(K/\boldsymbol{Q})=\sharp Ш(K/\boldsymbol{Q}).$$

In both Cases 1 and 2, we obtain

Theorem 2. $E(K/\boldsymbol{Q})=\sharp Ш(K/\boldsymbol{Q})$
where $Ш(K/\boldsymbol{Q})$ *is given by Theorem* 1.

By Corollary 1, we have

Corollary 2. $E(K/\boldsymbol{Q})=1 \Longleftrightarrow H_k$ *is cyclic.*

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References

[1] K. S. Brown: Cohomology of Groups. Springer, G T M., 87 (1982).
[2] M. Morishita: On S-class number relations of algebraic tori in Galois extensions of global fields (1990) (preprint).
[3] T. Ono: On some class number relations for Galois extensions. Nagoya Math. J., **107**, 121–133 (1987).
[4] ——: A note on the Artin Map. II. Proc. Japan Acad., **66**A, 132–136 (1990).
[5] M. J. Razar: Central and genus class fields and the Hasse norm theorem. Comp. Math., **35**, 281–298 (1977).
[6] J. Tate: Global class field theory. Algebraic Number Theory (eds. J. W. S. Cassels and A. Fröhlich). Thompson Book Company, Washington D.C., pp. 162–203 (1967).
[7] H. Wada: A table of ideal class groups of imaginary quadratic fields. Proc. Japan. Acad., **46**, 401–403 (1970).