81. *An Improvement of Sufficient Conditions for Starlike Functions*

By Mamoru NUNOKAWA,*) Shigeyoshi OWA,**) Hitoshi SAITOH,***) and Koichiro OHTAKE*)

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1. Introduction. Let A_p denote the class of functions of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in N=\{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic in the unit disk $U=\{z: |z|<1\}$.

A function $f(z) \in A_p$ is said to be p-valently starlike in U if it satisfies

$$\operatorname{Re} \frac{zf'(z)}{f(z)}>0 \qquad (z \in U). \tag{1.2}$$

We denote by $S^*(p)$ the subclass of A_p consisting of all such functions, and by $S^*(1)=S^*$ when $p=1$.

For $f(z)$ in the class A_1 when $p=1$, Singh and Singh [4] have proved

Theorem A. *If $f(z) \in A_1$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right|<\frac{3}{2} \qquad (z \in U), \tag{1.3}$$

then $f(z) \in S^*$.

Also, Miller and Mocanu [2] have showed

Theorem B. *If $f(z) \in A_1$ satisfies*

$$\left|\frac{f''(z)}{f'(z)}\right|<2 \qquad (z \in U), \tag{1.4}$$

then $f(z) \in S^*$.

In the present paper, we derive an improvement of the above theorems as the special cases of our main result.

2. Main theorem. In order to show our main result, we need the following lemma due to Jack [1] (also, by Miller and Mocanu [3]).

Lemma. *Let $w(z)$ be regular in U with $w(0)=0$. If $|w(z)|$ attains its maximum value in the circle $|z|=r$ at a point z_0, then we can write*

$$z_0 w'(z_0)=kw(z_0),$$

where k is a real number and $k \geqq 1$.

Applying the above lemma, we prove

Theorem. *Let $q(z)=p+q_1 z+q_2 z^2+\cdots$ $(p \in N)$ be analytic in U. If $q(z)$ satisfies*

$$\left|q(z)+\frac{zq'(z)}{q(z)}-p\right|<\frac{\sqrt{2}}{8}(5p+4\sqrt{p}+4) \qquad (z \in U), \tag{2.1}$$

*) Department of Mathematics, University of Gunma.
**) Department of Mathematics, Kinki University.
***) Department of Mathematics, Gunma College of Technology.

then

$$\mathrm{Re}\{q(z)\}>0 \qquad (z\in U). \tag{2.2}$$

Proof. We define the function $w(z)$ by

$$q(z)=(\sqrt{p}+w(z))^2. \tag{2.3}$$

Then $w(z)$ is regular in U with $w(0)=0$. Making use of the logarithmic differentiations in both sides of (2.3), we obtain that

$$q(z)+\frac{zq'(z)}{q(z)}-p=2\sqrt{p}\,w(z)+w(z)^2+\frac{2zw'(z)}{\sqrt{p}+w(z)}. \tag{2.4}$$

If we suppose that there exists a point $z_0\in U$ such that

$$\max_{|z|\leqq|z_0|}|w(z)|=|w(z_0)|=\frac{\sqrt{2p}}{2},$$

then lemma gives that

$$z_0w'(z_0)=kw(z_0) \qquad (k\geqq 1).$$

Therefore, letting $w(z_0)=(\sqrt{2p}/2)e^{i\theta}$, we have

$$\left|q(z_0)+\frac{z_0q'(z_0)}{q(z_0)}-p\right|=|w(z_0)|\left|2\sqrt{p}+w(z_0)+\frac{2z_0w'(z_0)}{(\sqrt{p}+w(z_0))w(z_0)}\right| \tag{2.5}$$

$$=\frac{\sqrt{2p}}{2}\left|2\sqrt{p}+\frac{\sqrt{2p}}{2}e^{i\theta}+\frac{2k}{\sqrt{p}+(\sqrt{2p}/2)e^{i\theta}}\right|$$

$$\geqq\frac{\sqrt{2p}}{2}\left|2\sqrt{p}+\frac{\sqrt{2p}}{2}\cos\theta+\frac{2k(2\sqrt{p}+\sqrt{2p}\cos\theta)}{p(3+2\sqrt{2}\cos\theta)}\right|$$

$$\geqq\sqrt{2}\left(p+\frac{\sqrt{2}}{4}p\cos\theta+\frac{2+\sqrt{2}\cos\theta}{3+2\sqrt{2}\cos\theta}\right).$$

Noting that the function $g(t)$ defined by

$$g(t)=p+\frac{\sqrt{2}}{4}pt+\frac{2+\sqrt{2}\,t}{3+2\sqrt{2}\,t} \qquad (t=\cos\theta) \tag{2.6}$$

has the minimum for $t=(2-3\sqrt{p})/2\sqrt{2p}$, we conclude that

$$\left|q(z_0)+\frac{z_0q'(z_0)}{q(z_0)}-p\right|\geqq\frac{\sqrt{2}}{8}(5p+4\sqrt{p}+4), \tag{2.7}$$

which contradicts our condition (2.1). Thus $|w(z)|<\sqrt{2p}/2$ for all $z\in U$.

It follows from this fact and (2.3) that $\mathrm{Re}\{q(z)\}>0$ for all $z\in U$.

Taking $p=1$, we have

Corollary 1. *Let* $q(z)=1+q_1z+q_2z^2+\cdots$ *be analytic in* U.

If $q(z)$ *satisfies*

$$\left|q(z)+\frac{zq'(z)}{q(z)}\right|<\frac{13\sqrt{2}}{8} \qquad (z\in U), \tag{2.8}$$

then $\mathrm{Re}\{q(z)\}>0$ $(z\in U)$.

Corollary 2. *If* $f(z)\in A_p$ *satisfies*

$$\left|\frac{zf''(z)}{f'(z)}+1-p\right|<\frac{\sqrt{2}}{8}(5p+4\sqrt{p}+4) \qquad (z\in U), \tag{2.9}$$

then $f(z)\in S^*(p)$.

Proof. Letting $q(z)=zf'(z)/f(z)$ in the theorem, we have

$$q(z)+\frac{zq'(z)}{q(z)}=1+\frac{zf''(z)}{f'(z)}. \tag{2.10}$$

Therefore, $f(z)\in S^*(p)$ follows from the theorem.

Making $p=1$ in Corollary 2, we have

Corollary 3. *If $f(z)\in A_1$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right|<\frac{13\sqrt{2}}{8} \qquad (z\in U), \tag{2.11}$$

then $f(z)\in S^*$.

Remark. Since $13\sqrt{2}/8=2.298\cdots$, Corollary 3 is an improvement of Theorem A by Singh and Singh [4], and of Theorem B by Miller and Mocanu [2].

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