10. A Remark on Exponentially Bounded C-semigroups

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1. **Introduction.** Let X be a Banach space with norm $\|\cdot\|$. We denote by $B(X)$ the set of all bounded linear operators from X into itself.

Let C be an injective operator in $B(X)$. A family $\{S(t); t\geqq 0\}$ in $B(X)$ is called an *exponentially bounded C-semigroup* (hereafter abbreviated to *C-semigroup*) *on* X, if

(1.1) $S(s+t)C=S(s)S(t)$ for $s, t\geqq 0$ and $S(0)=C$,

(1.2) $S(\cdot): [0, \infty)\to X$ is continuous for $x\in X$,

(1.3) there are $M\geqq 0$ and $a\geqq 0$ such that $\|S(t)\|\leqq Me^{at}$ for $t\geqq 0$.

The *generator* A of a C-semigroup $\{S(t); t\geqq 0\}$ on X is defined by

$$(1.4)\qquad \begin{cases} D(A)=\{x\in X; \lim_{t\to 0+}(S(t)x-Cx)/t\in R(C)\} \\ Ax=C^{-1}\lim_{t\to 0+}(S(t)x-Cx)/t \quad \text{for } x\in D(A), \end{cases}$$

where $R(C)$ denotes the range of C. It is known ([6, Proposition 1.1]) that

(1.5) A is a closed linear operator in X and $A=C^{-1}AC$.

The purpose of this note is to prove

Theorem 1. *The following statements are equivalent.*

(I) *A is the generator of a C-semigroup on X.*

(II) (a_1) *A is a closed linear operator in X satisfying $C^{-1}AC=A$.*

(a_2) *There exists a Banach space Σ with norm $N(\cdot)$ such that $R(C)\subset\Sigma\subset X$, $\|x\|\leqq M_1N(x)$ for $x\in\Sigma$, $N(x)\leqq M_2\|C^{-1}x\|$ for $x\in R(C)$ and the part of A in Σ is the generator of a semigroup of class (C_0) on Σ, where M_i, $i=1, 2$, are nonnegative constants.*

Corollary 2. *Let A be a closed linear operator in X, $c\in\rho(A)$ (the resolvent set of A) and let $n\geqq 0$ be an integer. Then the following statements are equivalent.*

(I′) *A is the generator of an n-times integrated semigroup on X.*

(II′) *A is the generator of a C-semigroup on X with $C=R(c; A)^n$, where $R(c; A)=(c-A)^{-1}$.*

(III′) *There exists a Banach space Σ with norm $N(\cdot)$ such that $D(A^n)\subset\Sigma\subset X$, $\|x\|\leqq M_1N(x)$ for $x\in\Sigma$, $N(x)\leqq M_2\Sigma_{k=0}^{n}\|A^kx\|$ for $x\in D(A^n)$ and the part of A in Σ is the generator of a semigroup of class (C_0) on Σ, where M_i, $i=1, 2$, are nonnegative constants.*

This corollary improves upon [4, Corollary 5.3].

2. **Proofs.** Let $\{S(t); t\geqq 0\}$ be a C-semigroup on X satisfying (1.3) and let $b>a$. We define a linear subset Σ of X and a norm $N(\cdot)$ on Σ by

(2.1) $\Sigma=\{x\in X; C^{-1}S(t)x$ is continuous in $t\geqq 0$ and $\lim_{t\to\infty}e^{-bt}\|C^{-1}S(t)x\|=0\}$,

(2.2) $N(x)=\sup_{t\geqq 0}e^{-bt}\|C^{-1}S(t)x\|$ for $x\in\Sigma$, respectively. It is easy to see the following (2.3)–(2.6):

(2.3) $R(S(t))\subset\Sigma$ for $t\geqq 0$, in particular $R(C)\subset\Sigma$;

(2.4) Σ becomes a Banach space under the norm $N(\cdot)$;

(2.5) $\|x\|\leqq N(x)$ for $x\in\Sigma$ and $N(x)\leqq M\|C^{-1}x\|$ for $x\in R(C)$;

(2.6) Σ is invariant under $C^{-1}S(t)$ for $t\geqq 0$, and

$$C^{-1}S(s)C^{-1}S(t)x=C^{-1}S(s+t)x \quad \text{for } x\in\Sigma \quad \text{and} \quad s,\ t\geqq 0.$$

For each $t\geqq 0$ we define a linear operator $T(t)\colon \Sigma\to\Sigma$ by

$$T(t)x=C^{-1}S(t)x \qquad \text{for } x\in\Sigma.$$

Let A be the generator of $\{S(t);\ t\geqq 0\}$ and let A_Σ be the part of A in Σ. Then we have

Proposition 3. *$\{T(t);\ t\geqq 0\}$ is a semigroup of class (C_0) on the Banach space Σ satisfying $N(T(t)x)\leqq e^{bt}N(x)$ for $x\in\Sigma$ and $t\geqq 0$, and A_Σ is the generator of the semigroup $\{T(t);\ t\geqq 0\}$.*

Proof. Clearly, $T(0)=I|_\Sigma$ (the identity on Σ), $T(s+t)=T(s)T(t)$ for $s,\ t\geqq 0$ and $N(T(t)x)=\sup_{s\geqq 0}e^{-bs}\|C^{-1}S(s+t)x\|\leqq e^{bt}N(x)$ for $x\in\Sigma$ and $t\geqq 0$. Let $x\in\Sigma$. Since $e^{-bt}C^{-1}S(t)x$ is uniformly continuous in $t\geqq 0$, we obtain that $N(T(h)x-x)=\sup_{t\geqq 0}e^{-bt}\|C^{-1}S(t+h)x-C^{-1}S(t)x\|\leqq\sup_{t\geqq 0}\|e^{-b(t+h)}C^{-1}S(t+h)x-e^{-bt}C^{-1}S(t)x\|+(e^{bh}-1)N(x)\to 0$ as $h\to 0+$. Therefore $\{T(t);\ t\geqq 0\}$ is a semigroup of class (C_0) on Σ.

Let $\mathfrak{A}$ be the generator of the semigroup $\{T(t);\ t\geqq 0\}$. If $x\in D(\mathfrak{A})$, then $\|(C^{-1}S(t)x-x)/t-\mathfrak{A}x\|\leqq N((T(t)x-x)/t-\mathfrak{A}x)\to 0$ as $t\to 0+$, which implies that $x\in D(A)\cap\Sigma$ and $Ax=\mathfrak{A}x\in\Sigma$, i.e., $x\in D(A_\Sigma)$ and $A_\Sigma x=\mathfrak{A}x$. Therefore $\mathfrak{A}\subset A_\Sigma$. To show $D(A_\Sigma)\subset D(\mathfrak{A})$, let $x\in D(A)$ and $Ax\in\Sigma$. Since $S(t)z-Cz=A\int_0^t S(s)z\,ds$ and $AS(t)y=S(t)Ay$ for $t\geqq 0$, $z\in X$ and $y\in D(A)$ (see [6, Proposition 1.2] or [1, Lemmas 2.7 and 2.8]), we see that $S(t)x-Cx=\int_0^t S(s)Ax\,ds=C\int_0^t C^{-1}S(s)Ax\,ds$ and then $T(t)x-x=\int_0^t T(s)Ax\,ds$ for $t\geqq 0$. Since $T(\cdot)Ax\colon [0,\infty)\to\Sigma$ is continuous, we obtain $N((T(t)x-x)/t-Ax)\to 0$ as $t\to 0+$ which means $x\in D(\mathfrak{A})$. Therefore $D(A_\Sigma)\equiv\{x\in D(A)\cap\Sigma;\ Ax\in\Sigma\}\subset\{x\in D(A);\ Ax\in\Sigma\}\subset D(\mathfrak{A})$. Q.E.D.

Remark 4. 1) The argument above shows that $D(A_\Sigma)=\{x\in D(A);\ Ax\in\Sigma\}$.

2) $T(t)Cx=C^{-1}S(t)Cx=S(t)x$ for $x\in X$ and $t\geqq 0$, because of $R(C)\subset\Sigma$.

Proof of Theorem 1. By (1.5), (2.3), (2.5) and Proposition 3, (I) implies (II). To show that (II) implies (I), let A_Σ be the part of A in Σ and let $\{T(t);\ t\geqq 0\}$ be the semigroup of class (C_0) on Σ generated by A_Σ.

For each $t\geqq 0$ we define a linear operator $S(t)\colon X\to X$ by

$$S(t)x=T(t)Cx \qquad \text{for } x\in X.$$

Then we have

$$\|S(t)x\|\leqq M_1N(T(t)Cx)\leqq M_1Ke^{\omega t}N(Cx)\leqq KM_1M_2e^{\omega t}\|x\| \text{ for } x\in X \text{ and } t\geqq 0,$$

where K and ω are nonnegative constants such that $N(T(t)z)\leqq Ke^{\omega t}N(z)$ for $z\in\Sigma$ and $t\geqq 0$. Clearly, $S(\cdot)\colon [0,\infty)\to X$ is continuous for $x\in X$. Since A_Σ is the generator of the semigroup $\{T(t);\ t\geqq 0\}$ of class (C_0), it is known that $(\lambda-A_\Sigma)^{-1}z=\int_0^\infty e^{-\lambda t}T(t)z\,dt$ for $z\in\Sigma$ and $\lambda>\omega$. (For example, see [3, chapter XI].) Since $R(C)\subset\Sigma$ and $C|_\Sigma\in B(\Sigma)$, we obtain

(2.7) $(\lambda-A_\Sigma)^{-1}Cx=\int_0^\infty e^{-\lambda t}T(t)Cx\,dt$ for $x\in X$ and $\lambda>\omega$,

(2.8) $C(\lambda-A_\Sigma)^{-1}z=\int_0^\infty e^{-\lambda t}CT(t)z\,dt$ for $z\in\Sigma$ and $\lambda>\omega$.

Moreover we have

(2.9) $C(\lambda-A_\Sigma)^{-1}z=(\lambda-A_\Sigma)^{-1}Cz$ for $z\in\Sigma$ and $\lambda>\omega$.

In fact, let $z\in\Sigma$ and $\lambda>\omega$. From $A=C^{-1}AC$ and $R(C)\subset\Sigma$ it follows that $C(\lambda-A_\Sigma)^{-1}z\in D(A)\cap\Sigma$ and $AC(\lambda-A_\Sigma)^{-1}z=CA(\lambda-A_\Sigma)^{-1}z=CA_\Sigma(\lambda-A_\Sigma)^{-1}z=\lambda C(\lambda-A_\Sigma)^{-1}z-Cz\in\Sigma$. Therefore $C(\lambda-A_\Sigma)^{-1}z\in D(A_\Sigma)$ and $A_\Sigma C(\lambda-A_\Sigma)^{-1}z=\lambda C(\lambda-A_\Sigma)^{-1}z-Cz$, which implies (2.9). It follows from (2.7)–(2.9) that $\int_0^\infty e^{-\lambda t}(T(t)Cz-CT(t)z)\,dt=0$ for $z\in\Sigma$ and $\lambda>\omega$. By the uniqueness theorem for Laplace transforms we get

$$T(t)Cz=CT(t)z \qquad \text{for } z\in\Sigma \text{ and } t\geqq 0.$$

This implies that $S(s)S(t)x=T(s)CT(t)Cx=T(s+t)C^2x=S(s+t)Cx$ for $x\in X$ and $s,t\geqq 0$. Therefore $\{S(t);t\geqq 0\}$ is a C-semigroup on X.

Let B be the generator of the C-semigroup $\{S(t);t\geqq 0\}$ on X. It is known that $C^{-1}BC=B$ and $(\lambda-B)^{-1}Cx=\int_0^\infty e^{-\lambda t}S(t)x\,dt$ for $x\in X$ and $\lambda>\omega$. (See [6, Propositions 1.1 and 1.2] or [1, Lemma 2.9].) It follows from (2.7) that

(2.10) $(\lambda-A_\Sigma)^{-1}Cx=(\lambda-B)^{-1}Cx$ for $x\in X$ and $\lambda>\omega$.

Hence $(\lambda-B)^{-1}C(\lambda-A)x=(\lambda-A_\Sigma)^{-1}C(\lambda-A)x=(\lambda-A_\Sigma)^{-1}(\lambda-A_\Sigma)Cx=Cx$ for $x\in D(A)$ and $\lambda>\omega$, which implies $Ax=C^{-1}BCx=Bx$ for $x\in D(A)$, i.e., $A\subset B$. (We have used here that $A=C^{-1}AC$ and $C(D(A))\subset D(A_\Sigma)$.) By (2.10) again, $(\lambda-A_\Sigma)^{-1}C(\lambda-B)x=(\lambda-B)^{-1}C(\lambda-B)x=Cx$ for $x\in D(B)$, which implies $Bx=C^{-1}ACx=Ax$ for $x\in D(B)$, i.e., $B\subset A$. Thus A is the generator of the C-semigroup $\{S(t);t\geqq 0\}$. Q.E.D.

Proof of Corollary 2. By [2, Theorem 2.4], (I′) is equivalent to (II′). To show that (II′) is equivalent to (III′), we use Theorem 1 with $C=R(c;A)^n$. By $AR(c;A)x=R(c;A)Ax$ for $x\in D(A)$, we see that $A\subset C^{-1}AC$. Next if $x\in D(C^{-1}AC)$, then $Cx=R(c;A)C(cx-C^{-1}ACx)=CR(c,A)(cx-C^{-1}ACx)$ and hence $x=R(c;A)(cx-C^{-1}ACx)\in D(A)$. Therefore we obtain $A=C^{-1}AC$. Moreover, $x\to\|(c-A)^n x\|(=\|C^{-1}x\|)$ defines a norm on $D(A^n)$ which is equivalent to the graph norm $\sum_{k=0}^n\|A^k x\|$ on $D(A^n)$. The result follows from Theorem 1. Q.E.D.

3. Application.

We start with

(a_1) *Representation of C-semigroups.* Let $\{S(t);t\geqq 0\}$ be a C-semigroup on X. If A is the generator of $\{S(t);t\geqq 0\}$ then

(3.1) $S(t)x=\lim_{n\to\infty}(1-tA/n)^{-n}Cx=\lim_{\lambda\to\infty}e^{-\lambda t}\sum_{n=0}^\infty \dfrac{t^n\lambda^{2n}(\lambda-A)^{-n}}{n!}Cx$

for $x \in X$ and the limit is uniform in t on every bounded interval. In particular, if $R(C)$ is dense in X then we have [5, Theorems 1.2 and 1.3].

In fact, Let Σ, $N(\cdot)$, $T(t)$ and A_Σ be as in Proposition 3. By the theory of semigroups of class (C_0), $T(t)$ can be represented as follows (see [3] or [8]): For every $z \in \Sigma$, $T(t)z = N(\cdot)-\lim_{n\to\infty}(1-tA_\Sigma/n)^{-n}z = N(\cdot)-\lim_{\lambda\to\infty} e^{-\lambda t}\sum_{n=0}^{\infty}(t^n\lambda^{2n}(\lambda-A_\Sigma)^{-n}/n!)z$ $(=N(\cdot)-\lim_{\lambda\to\infty}\exp(t\lambda A_\Sigma(\lambda-A_\Sigma)^{-1})z)$ uniformly in $t\geqq 0$ on every bounded interval, where $N(\cdot)$-lim means the limit with respect to $N(\cdot)$-norm. Noting $T(t)Cx = S(t)x$ for $x \in X$ and $t \geqq 0$ (see Remark 4), we obtain (3.1). If $R(C)$ is dense in X, then $(\lambda-\bar{G})^{-n}Cx = (\lambda-A)^{-n}Cx$ for $x \in X$, $\lambda > a$ and $n \geqq 0$, where $\bar{G}$ is the c.i.g. of $\{S(t); t\geqq 0\}$. Therefore [5, Theorems 1.2 and 1.3] follows from (3.1).

(a_2) *The abstract Cauchy problem.* Let A be the generator of a C-semigroup $\{S(t); t\geqq 0\}$ on X satisfying (1.3). Then for every $x \in D(A_\Sigma)$, $u(t, x) = C^{-1}S(t)x$ is a unique solution to the abstract Caucy problem

(ACP; A, x) $\quad (d/dt)u(t, x) = Au(t, x)$ for $t \geqq 0$ and $u(0, x) = x$.

In fact, let $T(t)$ and A_Σ be as in Proposition 3. The conclusion follows from the fact that $T(t)x$ is a unique solution to (ACP; A_Σ, x) for $x \in D(A_\Sigma)$ by the theory of semigroups of class (C_0).

Since $(\lambda-A)^{-1}C(X) \subset D(A_\Sigma)$ for $\lambda > a$, the result above improves upon [7, Corollary 1.3]. (We note here that $C(D(A)) \subset (\lambda-A)^{-1}C(X)$ and that $C(D(A)) = (\lambda-A)^{-1}C(X)$ if and only if $\lambda \in \rho(A)$.)

(a_3) *Generation of C-semigroups.* Applying Theorem 1 we can prove the following generation theorem of a C-semigroup (see [6, Theorem 2.1]): Let A be a densely defined closed linear operator in X such that $\lambda - A$ is injective, $D((\lambda-A)^{-m}) \supset R(C)$, $\|(\lambda-A)^{-m}C\| \leqq M/(\lambda-a)^m (\lambda > a, m \geqq 1)$ and $C^{-1}AC = A$. Then A is the generator of a C-semigroup on X.

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