80. *An Application of a Certain Fractional Derivative Operator*

By Shigeyoshi OWA*) and H. M. SRIVASTAVA**)

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The object of the present paper is to introduce and study a linear operator $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}$ which is defined in terms of a certain fractional derivative operator. Various interesting properties of the operator $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}$, including its connection with the Carlson-Shaffer operator $\mathcal{L}(a, c)$, are given. It is also shown how these operators can be applied successfully with a view to proving a number of inclusion and connection theorems involving starlike, convex, and prestarlike functions in the open unit disk $\mathcal{U}$.

1. Introduction. Let $\mathcal{A}$ be the class of functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the *open* unit disk

$$\mathcal{U}=\{z:|z|<1\}.$$

A function $f(z)\in\mathcal{A}$ is said to be *starlike of order* α if it satisfies the inequality:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \tag{1.2}$$

for some α $(0\leqq\alpha<1)$ and for all $z\in\mathcal{U}$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α.

Furthermore, a function $f(z)\in\mathcal{A}$ is said to be *convex of order* α if it satisfies the inequality:

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \tag{1.3}$$

for some α $(0\leqq\alpha<1)$ and for all $z\in\mathcal{U}$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions which are convex of order α.

Throughout this paper, it should be understood that functions such as

$$\frac{zf'(z)}{f(z)} \quad\text{and}\quad \frac{zf''(z)}{f'(z)},$$

which have *removable singularities* at $z=0$, have had these singularities removed in statements like (1.2) and (1.3).

It follows readily from (1.2) and (1.3) that (cf. Duren [2, p. 43, Theorem 2.12] for the special case $\alpha=0$)

$$f(z)\in\mathcal{K}(\alpha)\Longleftrightarrow zf'(z)\in\mathcal{S}^*(\alpha). \tag{1.4}$$

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*) Department of Mathematics, Kinki University, Higashi-Osaka, Osaka, Japan.

**) Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia, V8W 3P4, Canada.

For the functions $f_j(z)$ defined by

$$f_j(z)=\sum_{n=0}^{\infty} a_{j,n+1}z^{n+1} \qquad (j=1,2), \tag{1.5}$$

we denote by $f_1*f_2(z)$ the Hadamard product or convolution of the functions $f_1(z)$ and $f_2(z)$, that is,

$$f_1*f_2(z)=\sum_{n=0}^{\infty} a_{1,n+1}a_{2,n+1}z^{n+1}. \tag{1.6}$$

With a view to introducing the Carlson-Shaffer operator $\mathcal{L}(a,c)$, we define the function $\varphi(a,c;z)$ by

$$\varphi(a,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}z^{n+1} \qquad (c\neq 0,-1,-2,\cdots;\ z\in\mathcal{U}), \tag{1.7}$$

where $(\lambda)_n$ is the Pochhammer symbol given by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1, & \text{if } n=0,\\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n=1,2,3,\cdots.\end{cases} \tag{1.8}$$

Clearly, the function $\varphi(a,c;z)$ is an incomplete Beta function with

$$\varphi(a,c;z)=zF(1,a;c;z) \tag{1.9}$$

in terms of the Gaussian hypergeometric function $F(\alpha,\beta;\gamma;z)$ defined by

$$F(\alpha,\beta;\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!} \qquad (c\neq 0,-1,-2,\cdots;\ z\in\mathcal{U}). \tag{1.10}$$

Making use of the function $\varphi(a,c;z)$, Carlson and Shaffer [1] introduced a linear operator $\mathcal{L}(a,c)$ on $\mathcal{A}$ by the convolution:

$$\mathcal{L}(a,c)f(z)=\varphi(a,c;z)*f(z) \qquad (f(z)\in\mathcal{A}). \tag{1.11}$$

We observe that $\mathcal{L}(a,c)$ maps $\mathcal{A}$ onto itself. Moreover, if

$$a\neq 0,-1,-2,\cdots,$$

then $\mathcal{L}(c,a)$ is an inverse of $\mathcal{L}(a,c)$. Note also that (cf. [3, p. 1067])

$$\mathcal{K}(\alpha)=\mathcal{L}(1,2)\mathcal{S}^*(\alpha) \qquad (0\leqq\alpha<1) \tag{1.12}$$

and

$$\mathcal{S}^*(\alpha)=\mathcal{L}(2,1)\mathcal{K}(\alpha) \qquad (0\leqq\alpha<1). \tag{1.13}$$

Next we introduce the operator $\mathcal{N}_{0,z}^{\alpha,\beta,\eta}$, which is related rather closely to the fractional differential operator $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}$ considered by Sohi [5]. Indeed we have

$$\mathcal{N}_{0,z}^{\alpha,\beta,\eta}f(z)=\frac{\Gamma(2-\beta)\Gamma(3-\alpha+\eta)}{\Gamma(3-\beta+\eta)}z^{\beta}\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z) \qquad (f(z)\in\mathcal{A}), \tag{1.14}$$

where the fractional differential operator $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}$ is defined (for real numbers α, β, and η) by (see also [6])

$$\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\left\{z^{\alpha-\beta}\int_0^z(z-\zeta)^{-\alpha}\times F\left[\beta-\alpha,-\eta,1-\alpha;\ 1-\frac{\zeta}{z}\right]f(\zeta)d\zeta\right\} \tag{1.15}$$

$$(0\leqq\alpha<1;\ \varepsilon>\max\{0,\beta-\eta-1\}-1),$$

$f(z)$ being an analytic function in a simply-connected region of the z-plane containing the origin, with the order

$$f(z)=0(|z|^{\varepsilon}) \qquad (z\longrightarrow 0),$$

and the multiplicity of $(z-\zeta)^{-\alpha}$ being removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

In this paper we present several interesting properties and characteristics of the linear operator $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}$ and apply this operator in conjunction with the Carlson-Shaffer operator $\mathcal{L}(a, c)$ to prove a number of inclusion and connection theorems involving, for example, the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$.

2. Properties of the linear operator $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}$. We begin by proving an interesting relationship between the operators $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}$ and $\mathcal{L}(a, c)$, which is contained in

Lemma 1. *If* $0\leqq\alpha<1$ *and* $\beta-\eta<3$, *then*

$$\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)=\mathcal{L}(2, 2-\beta)\mathcal{L}(3-\beta+\eta, 3-\alpha+\eta)f(z) \qquad (f(z)\in\mathcal{A}). \tag{2.1}$$

Proof. It follows from the definition (1.15) that

$$\mathcal{I}_{0,z}^{\alpha,\beta,\eta}z^k=\frac{\Gamma(k+1)\Gamma(k+2-\beta+\eta)}{\Gamma(k+1-\beta)\Gamma(k+2-\alpha+\eta)}z^{k-\beta} \qquad (k+2>\beta-\eta), \tag{2.2}$$

which, in view of (1.1), yields

$$\begin{aligned}\mathcal{J}_{0,z}^{\alpha,\beta,\eta}f(z)&=z+\sum_{n=2}^{\infty}\frac{(2)_{n-1}(3-\beta+\eta)_{n-1}}{(2-\beta)_{n-1}(3-\alpha+\eta)_{n-1}}z^n\\&=\sum_{n=0}^{\infty}\frac{(2)_n(3-\beta+\eta)_n}{(2-\beta)_n(3-\alpha+\eta)_n}z^{n+1}\\&=\mathcal{L}(2, 2-\beta)\mathcal{L}(3-\beta+\eta, 3-\alpha+\eta)f(z),\end{aligned} \tag{2.3}$$

where we have also employed the definition (1.11).

Next we recall the following lemma due essentially to Carlson and Shaffer [1], which will be required in our present investigation (see also Owa and Srivastava [3, p. 1067, Remark 6]).

Lemma 2. *If* $\alpha\leqq\beta<1$ *and* $0\leqq\alpha<1$, *then*

$$\mathcal{L}(2-2\beta, 2-2\alpha)\mathcal{S}^*(\alpha)\subset\mathcal{S}^*(\beta)\subset\mathcal{S}^*(\alpha). \tag{2.4}$$

Making use of Lemma 1 and Lemma 2, we now prove an interesting inclusion property of the operators $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}$ and $\mathcal{L}(a, c)$. We first state our result as

Theorem 1. *If* $0\leqq\alpha<1$, $\beta-\eta<3$, *and* $0\leqq\beta<1$, *then*

$$\mathcal{L}(3-\alpha+\eta, 3-\beta+\eta)\mathcal{J}_{0,z}^{\alpha,\beta,\eta}\mathcal{K}\left[\frac{1}{2}\right]\subset\mathcal{S}^*\left[\frac{1}{2}\right]. \tag{2.5}$$

Proof. It is easy to see that, for $0\leqq\gamma<1$,

$$\begin{aligned}\mathcal{J}_{0,z}^{\alpha,\beta,\eta}\mathcal{K}(\gamma)&=\mathcal{L}(2, 2-\beta)\mathcal{L}(3-\beta+\eta, 3-\alpha+\eta)\mathcal{K}(\gamma)\\&=\mathcal{L}(2, 2-\beta)\mathcal{L}(3-\beta+\eta, 3-\alpha+\eta)\mathcal{L}(1, 2)\mathcal{S}^*(\gamma)\\&=\mathcal{L}(1, 2-\beta)\mathcal{L}(3-\beta+\eta, 3-\alpha+\eta)\mathcal{S}^*(\gamma).\end{aligned}$$

Therefore, we have

$$\mathcal{L}(3-\alpha+\eta, 3-\beta+\eta)\mathcal{J}_{0,z}^{\alpha,\beta,\eta}\mathcal{K}(\gamma)=\mathcal{L}(1, 2-\beta)\mathcal{S}^*(\gamma). \tag{2.6}$$

Since

$$\mathcal{S}^*\left[\frac{1}{2}\right]\subset\mathcal{S}^*\left[\frac{\beta}{2}\right] \qquad (0\leqq\beta<1), \tag{2.7}$$

we have

$$\mathcal{L}(1, 2-\beta)\mathcal{S}^*\left[\frac{1}{2}\right] \subset \mathcal{L}(1, 2-\beta)\mathcal{S}^*\left[\frac{\beta}{2}\right] \qquad (0 \leqq \beta < 1). \tag{2.8}$$

Thus, by an application of Lemma 2, we obtain

$$\mathcal{L}(1, 2-\beta)\mathcal{S}^*\left[\frac{\beta}{2}\right] \subset \mathcal{S}^*\left[\frac{1}{2}\right] \subset \mathcal{S}^*\left[\frac{\beta}{2}\right]. \tag{2.9}$$

Finally, on setting $\gamma = 1/2$ in (2.6), we complete the proof of Theorem 1.

3. An application involving prestarlike functions. A function $f(z) \in \mathcal{A}$ is said to be *prestarlike of order* α ($\alpha \leqq 1$) if and only if

$$\begin{cases} \dfrac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in \mathcal{S}^*(\alpha) & (\text{for } \alpha < 1) \\ \operatorname{Re}\left\{\dfrac{f(z)}{z}\right\} > \dfrac{1}{2} & (\text{for } \alpha = 1). \end{cases} \tag{3.1}$$

We denote by $\mathcal{R}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all prestarlike functions of order α. The class $\mathcal{R}(\alpha)$ was first introduced by Ruscheweyh [4].

In view of the definition (3.1) for the class $\mathcal{R}(\alpha)$, we have

$$\mathcal{R}(\alpha) = \mathcal{L}(1, 2-2\alpha)\mathcal{S}^*(\alpha) \qquad (\alpha < 1) \tag{3.2}$$

and

$$\mathcal{R}(1) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{2} \qquad (z \in \mathcal{U}) \right\}. \tag{3.3}$$

The following result provides a connection theorem involving the classes $\mathcal{K}(\alpha)$ and $\mathcal{R}(\alpha)$.

Theorem 2. *If* $0 \leqq \alpha < 1$, $\beta - \eta < 3$, *and* $0 \leqq \beta < 2$, *then*

$$\mathcal{L}(3-\alpha+\eta, 3-\beta+\eta)\mathcal{N}_{0,z}^{\alpha,\beta,\eta}\mathcal{K}\left[\frac{\beta}{2}\right] = \mathcal{R}\left[\frac{\beta}{2}\right]. \tag{3.4}$$

Proof. Since

$$\begin{aligned} \mathcal{N}_{0,z}^{\alpha,\beta,\eta}\mathcal{K}\left[\frac{\beta}{2}\right] &= \mathcal{L}(2, 2-\beta)\mathcal{L}(3-\beta+\eta, 3-\alpha+\eta)\mathcal{S}^*\left[\frac{1}{2}\right] \\ &= \mathcal{L}(1, 2-\beta)\mathcal{L}(3-\beta+\eta, 3-\alpha+\eta)\mathcal{S}^*\left[\frac{\beta}{2}\right], \end{aligned}$$

we obtain

$$\begin{aligned} \mathcal{L}(3-\alpha+\eta, 3-\beta+\eta)\mathcal{N}_{0,z}^{\alpha,\beta,\eta}\mathcal{K}\left[\frac{\beta}{2}\right] &= \mathcal{L}(1, 2-\beta)\mathcal{S}^*\left[\frac{\beta}{2}\right] \\ &= \mathcal{R}\left[\frac{\beta}{2}\right] \qquad (0 \leqq \beta < 2), \end{aligned}$$

which proves the assertion (3.4) of Theorem 2.

Taking $\beta = 0$ in Theorem 2, we have

Corollary 1. *If* $0 \leqq \alpha < 1$ *and* $\eta > -3$, *then*

$$\mathcal{L}(3-\alpha+\eta, 3+\eta)\mathcal{N}_{0,z}^{\alpha,0,\eta}\mathcal{K}(0) = \mathcal{R}(0). \tag{3.5}$$

Finally, setting $\beta = 1$ in Theorem 2, we deduce

Corollary 2. *If* $0 \leqq \alpha < 1$ *and* $\eta > -2$, *then*

$$\mathcal{L}(3-\alpha+\eta, 2+\eta)\mathcal{N}_{0,z}^{\alpha,1,\eta}\mathcal{K}\left[\frac{1}{2}\right] = \mathcal{R}\left[\frac{1}{2}\right]. \tag{3.6}$$

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