79. On Products of Consecutive Integers

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1. Diophantine equations involving products of integers have been investigated by many mathematicians. Among these are Erdös [1], L. J. Mordell [4], but these are the equations in two variables. In this paper, we shall show the following diophantine equation in three variables

(1) $$x(x+1)y(y+1)=z(z+1)$$

has infinitely many integer solutions and also show there exists an algorithm for obtaining all the integer solutions of (1).

2. In our previous paper [3], we have obtained the following result. We denote the set of all the integer solutions of a diophantine equation $z^2=(x^2-1)(y^2-1)+a$ $(a \in Z)$ by S_a. Then it is easy to verify that the mappings

$$\sigma: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ xy+z \\ (x^2-1)y+xz \end{pmatrix}, \qquad \tau: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ x \\ z \end{pmatrix},$$

$$\rho_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ y \\ z \end{pmatrix}, \quad \rho_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ -y \\ z \end{pmatrix}, \quad \rho_3: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$

are the permutations of S_a. G denotes the group $\langle \sigma, \tau, \rho_i \rangle$ $(1 \leqq i \leqq 3)$. We denotes the number of the representatives $\#[S_a/G]$ by t_a. Then we have obtained the following proposition.

Proposition (cf. [3]). *The number t_a is finite except in case $a=0$, and t_a equals the number of the integer points contained in the set $S_a \cap R_a$, where*

$$R_a = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 0 \leqq x \leqq y \leqq \sqrt{(a+1-x^2)/(2x+2)},\ 0 \leqq z \right\} \qquad \textit{in case } a>0,$$

and

$$R_a = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 1 < x \leqq y,\ 0 \leqq z,\ \sqrt{(x^2-a-1)/(x^2-1)} \leqq y \leqq \sqrt{(x^2-a-1)/(2x-2)} \right\}$$

in case $a<0$.

For the case $a=4$, we have $t_a=2$, that is, $S_a = G\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cup G\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Congruence consideration shows $G\begin{pmatrix}1\\1\\2\end{pmatrix}=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in S_a:\ x\equiv y\equiv 1 \bmod 2 \text{ and } z\equiv 2 \bmod 4\right\}$, which will be denoted by S^*. We denote by T the set of all the integer solutions of (1). Then the mapping $f:\begin{pmatrix}x\\y\\z\end{pmatrix}\to\begin{pmatrix}2x+1\\2y+1\\4z+2\end{pmatrix}$ is a bijection from T to S^*. We denote $\sigma^*=f^{-1}\sigma f$, $\tau^*=f^{-1}\tau f$ and $\rho_i^*=f^{-1}\rho_i f$. Then we have

$$\sigma^*:\begin{pmatrix}x\\y\\z\end{pmatrix}\longrightarrow\begin{pmatrix}x\\2xy+x+y+2z+1\\2x^2y+x^2+2xy+2xz+2x+z\end{pmatrix},$$

$$\tau^*:\begin{pmatrix}x\\y\\z\end{pmatrix}\longrightarrow\begin{pmatrix}y\\x\\z\end{pmatrix},\qquad \rho_1^*:\begin{pmatrix}x\\y\\z\end{pmatrix}\longrightarrow\begin{pmatrix}-x-1\\y\\z\end{pmatrix},$$

$$\rho_2^*:\begin{pmatrix}x\\y\\z\end{pmatrix}\longrightarrow\begin{pmatrix}x\\-y-1\\z\end{pmatrix}\quad\text{and}\quad \rho_3^*:\begin{pmatrix}x\\y\\z\end{pmatrix}\longrightarrow\begin{pmatrix}x\\y\\-z-1\end{pmatrix}.$$

G^* denotes the group $\langle\sigma^*,\tau^*,\rho_i^*\rangle$ $(1\leqq i\leqq 3)$. Then we have $T=G^*\begin{pmatrix}0\\0\\0\end{pmatrix}$, that is, G^* acts on T transitively. For example, a solution $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ is obtained by $\begin{pmatrix}1\\2\\3\end{pmatrix}=\sigma^*\tau^*\sigma^*\begin{pmatrix}0\\0\\0\end{pmatrix}$. Combining these, we have the following theorem.

Theorem. *With the above notation, we have* $T=G^*\begin{pmatrix}0\\0\\0\end{pmatrix}$.

References

[1] P. Erdös: Note on products of consecutive integers. J. London Math. Soc., **14**, 194–198 (1939).
[2] K. Kashihara: The diophantine equation $x^2-1=(y^2-1)(z^2-1)$. Mem. Anan College of Tech., **26**, 119–130 (1990) (in Japanese).
[3] S. Katayama and K. Kashihara: On the structure of the integer solutions of $z^2=(x^2-1)(y^2-1)+a$ (to appear in J. Math. Tokushima Univ., **24**).
[4] L. J. Mordell: On the integer solutions of $y(y+1)=x(x+1)(x+2)$. Pacific J. Math., **13**, 1347–1351 (1963).
[5] ——: Diophantine Equations. Academic Press, London, New York (1969).