66. *Comparison Principle for Singular Degenerate Elliptic Equations on Unbounded Domains**)

By Moto-Hiko SATO
Department of Mathematics, Hokkaido University

(Communicated by Shokichi IYANAGA, M. J. A., Oct. 12, 1990)

1. Introduction. This paper, as a preliminary study for [5], announces a comparison principle for viscosity solutions of singular degenerate elliptic equations

(1) $u+F(x, u, \nabla u, \nabla^2 u)=0$ in Ω ($\nabla u=\mathrm{grad}\, u$, $\nabla^2 u$: Hessian)

where Ω is a domain (not necessarily bounded) in $\boldsymbol{R}^n$. A typical example is

$$\lambda u-|\nabla u|\,\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)=0 \quad (\lambda \in \boldsymbol{R}). \tag{2}$$

This equation is derived from the mean curvature flow equation

$$v_t-|\nabla v|\,\mathrm{div}\left(\frac{\nabla v}{|\nabla v|}\right)=0 \tag{2'}$$

by setting $v(t, x)=e^{\lambda t}u(x)$; see also [1, 4]. The idea of the proof applies to parabolic equation in [5], so we omit the detailed proof since it is easily seen from the argument in [5]. The comparison principle for viscosity solutions is established by M.G. Crandall and P.L. Lions [3] for first order equations, by P.L. Lions [12], R. Jensen [10], H. Ishii [7] for second order degenerate elliptic equations (see also [11]), by Y.-G. Chen, Y. Giga and S. Goto [1] for singular parabolic equations including the mean curvature flow equations (see also [4]). However so far no results applied for (2) in an unbounded domain.

2. Comparison principle. Let Ω be a domain in $\boldsymbol{R}^n$ not necessarily bounded. We consider a degenerate elliptic equation of form

(3) $u+F(x, u, \nabla u, \nabla^2 u)=0$ in Ω.

In this paper we call a continuous function m: $[0, \infty)\to[0, \infty)$ a *modulus* if $m(0)=0$ and it is nondecreasing. We first list assumptions on $F=F(x, r, p, X)$.

(F1) F: $J(\Omega)=\Omega\times\boldsymbol{R}\times(\boldsymbol{R}^n\setminus\{0\})\times S^n\to\boldsymbol{R}$ is continuous, where S^n denotes the space of real $n\times n$ symmetric matrices.

(F2) F is *degenerate elliptic*, i.e., $F(x, r, p, X+Y)\le F(x, r, p, X)$ in $J(\Omega)$ if $Y\ge O$.

(F3) $-\infty<F_*(x, r, 0, O)=F^*(x, r, 0, O)<\infty$ for all $(x, r)\in\Omega\times\boldsymbol{R}$, where F_* and F^* are, respectively, *the lower and upper semicontinuous relaxation* (*envelope*) of F on $J(\bar{\Omega})$, i.e.,

*) In the memory of Professor Kôsaku YOSIDA, M. J. A.

$$F_*(x, r, p, X)=\liminf_{\varepsilon \downarrow 0} \{F(y, s, q, Y): q\neq 0, |x-y|\leq\varepsilon, |r-s|\leq\varepsilon, |p-q|\leq\varepsilon, |X-Y|\leq\varepsilon\}$$

and $F^*=-(-F)_*$.

Here $|X|$ denotes the operator norm of X as a selfadjoint operator on $\boldsymbol{R}^n$.

We assume that F is uniformly bounded in a neighborhood of $p=0$.

(F4) For every $R>0$

$$C_R=\sup\{|F(x, r, p, X)|: |p|, |X|\leq R, (x, r, p, X)\in J(\Omega), p\neq 0\}$$

is finite.

We assume a kind of monotonicity in r.

(F5) $r\mapsto F(x, r, p, X)$ is nondecreasing for all $(x, r, p, X)\in J(\Omega)$.

(F6) For every $R>\rho>0$ there is a modulus $\sigma=\sigma_{R_\rho}$ such that $|F(x, r, p, X)-F(x, r, q, Y)|\leq\sigma_{R_\rho}(|p-q|+|X-Y|)$ for all $(x, r)\in\Omega\times\boldsymbol{R}$, $\rho\leq|p|, |q|\leq R, |X|, |Y|\leq R$.

The behavior near $(p, X)=(0, O)$ is assumed to be uniform in x and r.

(F7) There are $\rho_0>0$ and a modulus σ_1 such that

$$F^*(x, r, p, X)-F^*(x, r, 0, O)\leq\sigma_1(|p|+|X|)$$
$$F_*(x, r, p, X)-F_*(x, r, 0, O)\geq-\sigma_1(|p|+|X|)$$

provided that $(x, r)\in\Omega\times\boldsymbol{R}$ and $|p|, |X|\leq\rho_0$.

(F8) There is a modulus σ_2 such that

$$|F(x, r, p, X)-F(y, r, p, X)|\leq\sigma_2(|x-y|(|p|+1))$$

for $y\in\Omega$, $(x, r, p, X)\in J(\Omega)$.

We shall also use the following weaker assumptions in place of (F2), (F6), (F7) and (F8).

(F9) For every $R>\rho>0$ there is a modulus $\sigma=\sigma_{R_\rho}$ such that

$$|F(x, r, p, X)-F(x, r, q, X)|\leq\sigma_{R_\rho}(|p-q|)$$

for all $y\in\Omega$, $(x, r, p, X)\in J(\Omega)$, $\rho\leq|p|, |q|\leq R, |X|\leq R$.

(F10) There is a modulus σ_2 such that

$$F_*(x, r, 0, O)-F^*(y, r, 0, O)\geq-\sigma_2(|x-y|)$$

for all $y\in\Omega$, $(x, r)\in\Omega\in\boldsymbol{R}$.

(F11) Suppose that

$$-\mu\begin{pmatrix} I & O\\ O & I\end{pmatrix}\leq\begin{pmatrix} X & O\\ O & Y\end{pmatrix}\leq\nu\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}+\omega\begin{pmatrix} I & O\\ O & I\end{pmatrix} \quad \text{with } \mu, \nu, \omega\geq 0.$$

Let R be taken so that $R\geq\max(\mu, \theta)+2\omega$ with $\theta=2\nu+\omega$. Let ρ be a positive number. Then it holds

$$F_*(x, r, p, X)-F^*(y, r, p, -Y)\geq-\bar{\sigma}(|x-y|(|p|+1)+\nu|x-y|^2)-\bar{\sigma}(2\omega)$$

for all $\rho\leq|p|\leq R$ with some modulus $\bar{\sigma}=\bar{\sigma}_{R_\rho}$ independent of $x, y, r, X, Y, \mu, \nu, \omega$.

Theorem 2.1. *Suppose that F satisfies* (F1)–(F8). *Let u and v be, respectively, sub- and supersolutions of* (3) *in Ω. Assume that*

(A1) *$u(x)\leq K(|x|+1)$, $v(x)\geq-K(|x|+1)$ for some $K>0$ independent of $x\in\Omega$;*

(A2) *there is a modulus m_0 such that $u^*(x)-v_*(y)\leq m_0(|x-y|)$ for all $(x, y)\in\partial(\Omega\times\Omega)$;*

(A3) *$u^*(x)-v_*(y)\leq K(|x-y|+1)$ on $\partial(\Omega\times\Omega)$ for some $K>0$ independent of $(x,y)\in\partial(\Omega\times\Omega)$.*

Then there is a modulus m such that

$$u^*(x)-v_*(y)\leq m(|x-y|) \quad \text{on } \Omega\times\Omega. \tag{4}$$

In particular $u^*\leq v_*$ on Ω.

Theorem 2.2. *Suppose that F satisfies* (F1), (F3)–(F5), (F9)–(F11). *Let u and v be, respectively, viscosity sub- and supersolutions of* (3) *in Ω. Assume that* (A1)–(A3) *hold for u and v. Then there is a modulus m such that* (4) *holds.*

Remark 2.1. The assumption (F8) has a disadvantage that it excludes variable coefficients in second order term. Theorem 2.1 is the special case of Theorem 2.2. Indeed the assumption (F3) and (F8) imply (F10) and (F2), (F6), (F8) imply (F11). If F is independent of x, the assumption (F2) is equivalent to the assumption (F11).

Example. Let $\Sigma(x,p)$ be a bounded function on $\bar{\Omega}\times(\boldsymbol{R}^n\setminus\{0\})$ with values in the space of $n\times n$ real matrices. Suppose that Σ is Lipschitz on $\bar{\Omega}\times\{p\in\boldsymbol{R}^n : |p|\geq\rho\}$ for every $\rho>0$ and that

$$F(x,p,X)=-\operatorname{trace}(\Sigma(x,p)^t\Sigma(x,p)X).$$

It is easy to see F satisfies all other assumptions in Theorem 2.2 although F may not satisfy (F8).

3. A sketch of the proof of Theorem 2.2. We prove Theorem 2.2 in several steps. We will note several propositions to prove Theorem 2.2. We begin by deriving a rough growth estimate for $u(x)-v(y)$ on Ω.

Proposition 3.1. *Suppose that* (F1) *and* (F4). *Let u and v be, respectively, viscosity sub- and supersolutions of* (3) *in Ω. Assume that u and $-v$ are upper semicontinuous in Ω. Moreover, assume that u and v satisfy* (A1) *and* (A3). *Then for $K'>K$ there is a constant $M=M(K',F)$ such that*

$$u(x)-v(y)\leq K'(|x-y|+1) \quad \text{on } \Omega\times\Omega. \tag{5}$$

For ε, $\delta>0$ we set

$$\Phi(x,y)=w(x,y)-\Psi(x,y), \qquad w(x,y)=u(x)-v(y),$$

$$\Psi(x,y)=\frac{|x-y|^4}{4\varepsilon}+B(x,y), \qquad B(x,y)=\delta(|x|^2+|y|^2).$$

The function B plays the role of a barrier for space infinity.

Proposition 3.2. *Suppose that u and v satisfy* (F5) *and that*

$$\alpha=\limsup_{\theta\downarrow 0}\{w(x,y) : |x-y|<\theta,\ (x,y)\in\Omega\times\Omega\}>0. \tag{6}$$

Then there are positive constants δ_0 such that

$$\sup_{\Omega\times\Omega}\Phi(x,y)>\frac{\alpha}{2} \tag{7}$$

holds for all $0<\delta<\delta_0$, $\varepsilon>0$.

Proposition 3.3. *Let u, v, δ_0 be as in Proposition* 3.2. *Suppose that w is upper semicontinuous in $\Omega\times\Omega$.*

(i) *Φ attains a maximum over $\bar{\Omega}\times\bar{\Omega}$ at $(\hat{x},\hat{y})\in\bar{\Omega}\times\bar{\Omega}$.*

(ii) *$|\hat{x}-\hat{y}|$ is bounded as a function of $0<\varepsilon<1$, $0<\delta<\delta_0$.*

(iii) *$\delta\hat{x}$ and $\delta\hat{y}$ tend to zero as $\delta\to 0$; the convergence is uniform in* $0<\varepsilon<1$.

In particular, for fixed $\delta>0$, $\hat{x}$ and $\hat{y}$ are bounded on $0<\varepsilon<1$.

(iv) *$|\hat{x}-\hat{y}|$ tends to zero as $\varepsilon\to 0$; the convergence is uniform in* $0<\delta<\delta_0$.

(v) $\lim_{\varepsilon\downarrow 0}\overline{\lim_{\delta\downarrow 0}}\dfrac{|\hat{x}-\hat{y}|^4}{\varepsilon}=0.$

Proposition 3.4. *Assume the hypotheses of Proposition* 3.3. *Suppose that* (A2) *holds for u and v. Then there is $\varepsilon_0>0$ such that Φ attains a maximum over $\bar{\Omega}\times\bar{\Omega}$ at an interior point $(\hat{x},\hat{y})\in\Omega\times\Omega$, i.e., $(\hat{x},\hat{y})\in\Omega\times\Omega$ for all* $0<\varepsilon<\varepsilon_0$, $0<\delta<\delta_0$.

Proof of Theorem 2.2. We argue by a contradiction. Suppose that (4) were false. Then we would have (6). By Proposition 3.1 and (6) we see all conclusions in Propositions 3.2–3.4 would hold. By Proposition 3.4 Φ attains a maximum over $\bar{\Omega}\times\bar{\Omega}$ at $(\hat{x},\hat{y})\in\Omega\times\Omega$ for small ε, δ. Therefore $w(x,y)\le w(\hat{x},\hat{y})+\Psi(x,y)-\Psi(\hat{x},\hat{y})$ in $\Omega\times\Omega$.

Expanding Ψ at $(\hat{x},\hat{y})$ yields

$$(\hat{\Psi}_{x,y},A)(\hat{x},\hat{y})\in J^{2,+}w(\hat{x},\hat{y})\quad\text{with}\quad \nabla^2\Psi(\hat{x},\hat{y})\le A$$

where $\hat{\Psi}_{x,y}=\nabla\Psi(\hat{x},\hat{y})$ and $\nabla=(\nabla_x,\nabla_y)$.

Here $J^{2,+}$ and $J^{2,-}$, respectively, denote *the sub- and super 2-jets* (see [2]). Since $(\hat{x},\hat{y})$ is an interior point of $\Omega\times\Omega$ and F satisfy (F4), we apply Theorem 1 in [2] and conclude that for each $\lambda>0$ there exists $X,Y\in S^n$ such that

$$(\hat{\Psi}_x,X)\in\bar{J}^{2,+}u(\hat{x}),\ (-\hat{\Psi}_y,-Y)\in\bar{J}^{2,-}v(\hat{y}) \tag{8}$$

$$-\Big(\frac{1}{\lambda}+|A|\Big)I\le\begin{pmatrix}X & O\\ O & Y\end{pmatrix}\le A+\lambda A^2, \tag{9}$$

where $\hat{\Psi}_x=\nabla_x\Psi(\hat{x},\hat{y})$, $\hat{\Psi}_y=\nabla_y\Psi(\hat{x},\hat{y})$. Here $\bar{J}^{2,+}$ and $\bar{J}^{2,-}$, respectively, denote the closure of $J^{2,+}$ and $J^{2,-}$. Since u and v are, respectively, *sub- and supersolutions* of (3), it follows from (8) that $\hat{u}+F_*(\hat{x},\hat{u},\hat{\Psi}_x,X)\le 0$, $\hat{v}+F^*(\hat{y},\hat{v},-\hat{\Psi}_y,-Y)\ge 0$ which yields

$$0\ge\hat{u}-\hat{v}+F_*(\hat{x},\hat{u},\hat{\Psi}_x,X)-F^*(\hat{y},\hat{v},-\hat{\Psi}_y,-Y), \tag{10}$$

where $\hat{u}=u(\hat{x})$, $\hat{v}=v(\hat{y})$. By (F5) this estimate yields

$$0\ge\alpha/2+F_*(\hat{x},\hat{u},\hat{\Psi}_x,X)-F^*(\hat{y},\hat{u},-\hat{\Psi}_y,-Y) \tag{11}$$

since Proposition 3.2 implies $\hat{u}-\hat{v}>\alpha/2$. We devide the situation in two cases depending on the behavior of $\hat{x}-\hat{y}$ as $\delta\to 0$. In both cases we have $\alpha/2\le 0$ considering Proposition 3.2, which yields a contradiction. We thus proved (4).

References

[1] Y.-G. Chen, Y. Giga and S. Goto: Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. Proc. Japan Acad., **65A**, 207–210 (1989) (Announcement); J. Diff. Geom. (to appear).

[2] M. G. Crandall and H. Ishii: The maximum principle for semicontinuous functions (preprint).

[3] M. G. Crandall and P. L. Lions: Viscosity solutions of Hamilton-Jacobi equations. Tran. AMS., **277**, 1–42 (1983).

[4] L. C. Evans and J. Spruck: Motion of level sets by mean curvature. I (to appear in J. Diff. Geom.).

[5] Y. Giga *et al.*: Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains (preprint).

[6] H. Ishii: Uniqueness of unbounded viscosity solution of Hamilton-Jacobi equations. Indiana Univ. Math. J., **33**, 721–748 (1984).

[7] ——: On uniqueness and existence of viscosity solutions of fully nonlinear second order elliptic PDE's. Comm. Pure Appl. Math., **42**, 15–45 (1989).

[8] H. Ishii and P. L. Lions: Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. J. Differential Equations, **83**, 26–78 (1990).

[9] R. Jensen: The maximum principle for viscosity solutions of fully nonlinear second-order partial differential equations. Arch. Rational Mech. Anan., **101**, 1–27 (1988).

[10] ——: Uniqueness criteria for viscosity solution of fully non linear elliptic partial differential equations. Indiana Univ. Math. J., **38**, 629–667 (1989).

[11] R. Jensen, P. L. Lions and P. E. Souganidis: A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations. Proc. AMS., **102**, 975–978 (1988).

[12] P. L. Lions: Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. II. Viscosity solutions and uniqueness. Comm. in P.D.E., **8**, 1229–1276 (1983).