64. *Yang-Mills Connections on Quaternionic Kähler Quotients*

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The purpose of this note is to announce our recent results on quaternionic Kähler manifolds (see Salamon [8] for definition of quaternionic Kähler manifolds). Let (M, g) be a $4n$-dimensional connected quaternionic Kähler manifold with scalar curvature s and let $\boldsymbol{H}$ be the skew field of quaternions $(\boldsymbol{H}=\boldsymbol{R}+\boldsymbol{R}i+\boldsymbol{R}j+\boldsymbol{R}k)$. Furthermore, let ρ be an $Sp(n)\cdot Sp(1)$-module induced by adjoint representation of $Sp(1)$. Then the vector bundle V corresponding to ρ is a subbundle in End(TM), whose rank is three. The Levi-Civita connection induces a metric connection on End(TM) naturally. The subbundle V is preserved by the connection, which is restricted to the connection on V, denoted by ∇. For each point in M, there are local frames I, J, K of V associated to $i, j, k \in \mathfrak{sp}(1) \subset \boldsymbol{H}$ on a neighbourhood of the point. We denote by ω_α $(\alpha=I, J, K)$, 2-forms $g(\alpha, \ )$ $(\alpha=I, J, K)$. Then $\sum_{\alpha \in I,J,K} \omega_\alpha \otimes \alpha$ defined locally can be globalized as a section on M to $\wedge^2 T^*M \otimes V$, which is denoted by $\Omega \in \Gamma(M, \wedge^2 T^*M \otimes V)$ (cf. [2]).

Let G be a compact Lie group which acts on M preserving the quaternionic Kähler structure g, V. Let $\mathfrak{g}$ be the Lie algebra of G.

Definition 1 (cf. [2], [5]). A section μ to $\mathfrak{g}^* \otimes V$ is a moment mapping for the action of G on M if

(i) $\nabla(\mu(X)) = \iota_{X^*}\Omega$, where X is an element of and X^* is the Killing vector field associated to X,

(ii) μ is a G-equivariant mapping.

When the scalar curvature s of M is not zero and G is connected, the moment mapping exists uniquely (see [2] for the proof). By the condition (ii), the set $\mu^{-1}(0)$ is G-invariant. Suppose that $\mu^{-1}(0)$ is a non-empty, submanifold in M and that G acts on it freely. Then the quotient $N=\mu^{-1}(0)/G$ is a manifold and g, V are naturally pushed down to the metric $\bar{g}$, the structure bundle $\bar{V}$ on N. The reduction $(N, \bar{g}, \bar{V})$ is a quaternionic Kähler manifold of dimension $4m = 4n - 4\dim(G)$ and it is called a quaternionic Kähler reduction (or hyperkähler reduction when $s=0$). Now we denote by

$$p: \mu^{-1}(0) \longrightarrow N$$

the principal bundle, which has a natural G-connection η as follows: the horizontal space is the orthogonal complement to the fibre with respect to g.

On the other hand, the $Sp(m)\cdot Sp(1)$-module $\wedge^2 H^m$ is a direct sum

$N_2'\oplus N_2''\oplus L_2$ of its irreducible submodules N_2', N_2'', L_2, where N_2' (resp. L_2) is the submodule fixed by $Sp(m)$ (resp. $Sp(1)$) and for $m=1$, we have $N_2''=\{0\}$. Hence the vector bundle $\wedge^2 T^*N$ is written as a direct sum $A_2'\oplus A_2''\oplus B_2$ of its holonomy invariant subbundle in such a way that A_2', A_2'', B_2 correspond to N_2', N_2'', L_2, respectively.

Let $q: Q\to N$ be a principal bundle whose fibre is a Lie group K ($\mathfrak{k}:=$ the Lie algebra).

Definition 2 (cf. [6]). A connection on $q: Q\to N$ is called a B_2-connection if the corresponding curvature is $\mathfrak{k}$-valued q^*B_2-form.

Now we obtain:

Theorem. *The connection η is a B_2-connection.*

Proof. The space $\mu^{-1}(0)$ is a submanifold in M. We denote the second fundamental form by π. By definition the Levi-Civita connection ∇_1 on $\mu^{-1}(0)$ is written as: for vector fields s, $w\in\mathcal{X}(\mu^{-1}(0))$

(1) $$\nabla^M_s w=\nabla_{1s}w+\pi(s, w),$$

where ∇^M is the Levi-Civita connection on (M, g). We denote by $\tilde{x}$ and w^v, the horizontal lift of $x\in\mathcal{X}(N)$ and the vertical component of $w\in\mathcal{X}(\mu^{-1}(0))$, i.e.

$$\eta(\tilde{x})=0, \qquad p_*(\tilde{x})=x,$$
$$\eta(w-w^v)=0.$$

By O'Neill's formula (cf. [7]) for Riemannian submersion, if x, $y\in\mathcal{X}(N)$,

(2) $$\widetilde{\nabla^N_x y}=\nabla_{1\tilde{x}}\tilde{y}-1/2[\tilde{x}, \tilde{y}]^v,$$

where ∇^N is the Levi-Civita connection on N. Equations (1), (2) lead to

(3) $$\widetilde{\nabla^N_s w}=\nabla^M_{\tilde{s}}\tilde{w}-\pi(\tilde{s}, \tilde{w})-1/2[\tilde{s}, \tilde{w}]^v.$$

For any point $n\in N$, there exists a local neighbourhood $n\in U\subset N$ such that the quaternionic structure bundle on N is spanned by I, J, K on U. When we exchange w to Iw

(4) $$\widetilde{\nabla^N_s Iw}=\nabla^M_{\tilde{s}}\widetilde{Iw}-\pi(\tilde{s}, \widetilde{Iw})-1/2[\tilde{s}, \widetilde{Iw}]^v, \qquad \text{on } U.$$

If we denote by $\bar{I}$, $\bar{J}$, $\bar{K}$ the pullback of I, J, K to TM on $\mu^{-1}(0)$, then

(5) $$\widetilde{Iw}=\bar{I}\tilde{w}.$$

Since M is a quaternionic Kähler manifold,

(6) $$\nabla^M\bar{I}=a_{12}\bar{J}+a_{13}\bar{K},$$

where a_{12}, a_{13} are connection forms with respect to the local frame I, J, K. We obtain by (4), (5), (6),

$$\bar{I}\nabla^M_{\tilde{s}}\tilde{w}+a_{12}(\tilde{s})\bar{J}\tilde{w}+a_{13}(\tilde{s})\bar{K}\tilde{w}$$
$$=\widetilde{\nabla^N_s Iw}+\pi(\tilde{s}, \bar{I}\tilde{w})+1/2[\tilde{s}, \bar{I}\tilde{w}]^v,$$

and by (3),

(7) $$\bar{I}\widetilde{\nabla^N_s w}+\bar{I}\pi(\tilde{s}, \tilde{w})+1/2\bar{I}[\tilde{s}, \tilde{w}]^v+a_{12}(\tilde{s})\bar{J}\tilde{w}+a_{13}(\tilde{s})\bar{K}\tilde{w}$$
$$=\widetilde{\nabla^N_s Iw}+\pi(\tilde{s}, \bar{I}\tilde{w})+1/2[\tilde{s}, \bar{I}\tilde{w}]^v.$$

The vertical component of (7) is

$$(\bar{I}\pi(\tilde{s}, \tilde{w}))^v=1/2[\tilde{s}, \bar{I}\tilde{w}]^v.$$

Since π is symmetric, we obtain:

$$(8)\qquad \begin{aligned}[\tilde{s}, \bar{I}\tilde{w}]^v &= 2(\bar{I}\pi(\tilde{s}, \tilde{w}))^v \\ &= 2(\bar{I}\pi(\tilde{w}, \tilde{s}))^v \\ &= [\tilde{w}, \bar{I}\tilde{s}]^v \\ &= -[\bar{I}\tilde{s}, \tilde{w}]^v.\end{aligned}$$

The curvature of η is written as $R(\tilde{s}, \tilde{w}) = -\eta([\tilde{s}, \tilde{w}]^v)$. By (8),

$$\begin{aligned}R(\widetilde{Is}, \widetilde{Iw}) &= -\eta([\widetilde{Is}, \widetilde{Iw}]^v) \\ &= -\eta(-[\tilde{s}, \widetilde{IIw}]^v) \\ &= -\eta([\tilde{s}, \tilde{w}]^v) \\ &= R(\tilde{s}, \tilde{w}).\end{aligned}$$

By same argument, $R(\widetilde{Is}, \widetilde{Iw}) = R(\widetilde{Js}, \widetilde{Jw}) = R(\widetilde{Ks}, \widetilde{Kw}) = R(\tilde{s}, \tilde{w})$. Hence the connection η is a B_2-connection.

Examples. (i) Galicki and Lawson proved the reduction space $p^n\boldsymbol{H}//U(1)$ is complex Grassmann manifold $G_{2,n-1}(\boldsymbol{C})$ (cf. [2]). The connection on $P \to G_{2,n-1}(\boldsymbol{C})$ is a B_2-connection. Furthermore Galicki showed the quotient space $P^n\boldsymbol{H}//SU(2)$ is real Grassmann manifold $G_{4,n-3}(\boldsymbol{R})$ (cf. [1]). It has also a B_2-connection.

(ii) The argument is local. When $\mu^{-1}(0)/G$ is not a smooth manifold but an orbifold, the connection is a B_2-connection over the orbifold. Galicki and Nitta constructed many quaternionic Kähler orbifolds as quaternionic Kähler reduction spaces (cf. [4]). In these cases the connections are B_2-connections over the quaternionic Kähler orbifolds.

Remark. A corresponding result for the case of hyperkähler reductions was previously obtained by Gocho and Nakajima [3]. Our result is inspired by their result.

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References

[1] K. Galicki: A generalization of the momentum mapping construction for quaternionic Kähler manifolds. Commun. Math. Phys., **108**, 117–138 (1987).
[2] K. Galicki and H. B. Lawson, Jr.: Quaternionic reduction and quaternionic orbifolds. Math. Ann., **282**, 1–21 (1988).
[3] T. Gocho and H. Nakajima: Einstein-Hermitian connections on hyperkähler quotients (to appear).
[4] K. Galicki and T. Nitta: Non-zero scalar curvature generalizations of the ALE hyperkähler metrics (to appear).
[5] N. J. Hitchin *et al.*: Hyperkähler metrics and supersymmetry. Comm. Math. Phys., **108**, 535–589 (1987).
[6] T. Nitta: Vector bundles over quaternionic Kähler manifolds. Tohoku Math. J., **40**, 425–440 (1988).
[7] B. O'Neill: The fundamental equations of submersion. Michigan Math. J., **13**, 459–469 (1966).
[8] S. Salamon: Quaternionic Kähler manifolds. Invent. Math., **67**, 143–171 (1982).