55. *The Mean Square of Dirichlet L-functions*

(*A Generalization of Balasubramanian's Method*)

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§ 1. The present article is a corrected version of the author's erroneous note [4]. (The report [5] includes the same errors. The corrections of the results have already been published in [6].) Let χ be a primitive Dirichlet character mod q, and $L(s,\chi)$ the corresponding Dirichlet L-function. Then,

Theorem. *For any odd q and any $T>0$, we have*

$$\int_0^T \left|L\left(\frac{1}{2}+it,\chi\right)\right|^2 dt=\frac{\varphi(q)}{q}T\log T+\frac{\varphi(q)}{q}\left(\log\frac{q}{2\pi}+2\gamma-1+2\sum_{p|q}\frac{\log p}{p-1}\right)T$$
$$+E(T,\chi),$$

where $\varphi(q)$ is Euler's function, γ is Euler's constant, p runs over all prime divisors of q, and

$$(1.1)\quad E(T,\chi)=O((qT)^{1/3+\varepsilon}+q(\log q)(qT)^{1/4}(\log(T+1))^{1/2}+q^{7/4}(\log q)(qT)^{1/4}$$
$$+q^{21/8}(\log q)(qT)^{1/8}+q^{13/4}(\log q)(\log q(T+1))^2+q^5(\log q)(\log q(T+1)))$$

for any $\varepsilon>0$.

In particular, if $T\gg q^{20}$, then $E(T,\chi)\ll(qT)^{1/3+\varepsilon}$. Motohashi [9] gives a better estimate in case q is a prime.

Only after the appearance of Motohashi's above mentioned work and the announcement (in Zentralblatt für Mathematik) of Meurman's paper [8], the author had noticed that the statement in [4]* is incorrect. Then the author started checking the former calculations, and found two essential errors. One of them is related with the Riemann-Siegel formula of L-functions (see (2) in [6]). Using the notations in [4], we can state it as

$$(1.2)\qquad e^{i\vartheta}L\left(\frac{1}{2}+it,\chi\right)=f_1(t)+f_2(t)+f_3(t),$$

and the correct estimate of the error term $f_3(t)$ is $O(q^{5/4}t^{-3/4})$. From (1.2), we have

$$(1.3)\quad \int_0^T\left|L\left(\frac{1}{2}+it,\chi\right)\right|^2 dt=\int_0^T f_1(t)^2dt+\int_0^T f_2(t)^2dt+\int_0^T f_3(t)^2dt$$
$$+2\int_0^T f_1(t)f_2(t)dt+2\int_0^T f_2(t)f_3(t)dt+2\int_0^T f_3(t)f_1(t)dt.$$

The other error in [4] is in the evaluation of the fourth integral in the right-hand side of (1.3). It is the reason of the appearance of the term $4(2\pi/q^3)^{1/2}(\Sigma'n)T^{1/2}$ in Theorem 1 of [4], which is to be omitted. A drastic change of the argument was required to correct this error, and the corrected calculations were completed in November 1989. Also, several points in former calculations were refined, so the error estimate (1.1) is slightly better than that stated in [4].

It should be emphasized that the first correct work which has obtained a $(qT)^{1/3}$-type estimate of $E(T,\chi)$ is Motohashi [9]. Meurman [8] has proved the asymptotic formula of $\sum_{\chi \bmod q}\int_0^T \left|L\left(\frac{1}{2}+it,\chi\right)\right|^2 dt$, also with a $(qT)^{1/3}$-type error term. Their methods, quite different from the author's, are based on Atkinson's idea [1]. It should be pointed out that our method cannot be applied to even q, while Motohashi's method is applicable to any modulus, though he has only published the result on the case q is a prime. In his unpublished manuscript [10], Motohashi indicates the way how to modify his argument so as to include the cases $q=1$ and composite q; in fact, [10] contains a detailed treatment of the relevant character sums for general modulus.

In the rest of this note we give a brief outline of the evaluation of the right-hand side of (1.3). Details are written in [7].

§ 2. Since our theorem is reduced to Balasubramanian's result [2] itself in case $q=1$, we can assume $q\geqq 3$. Let $n=(n_1-1)q+n_2$, $m=(m_1-1)q+m_2$ $(0<n_2<q,\ 0<m_2<q)$, $r=n-m$, $r_1=n_1-m_1$, $L=q[(T/2\pi q)^{1/2}]$, $K=L/q$ and

$$J=J(n_2,m_2)=\begin{cases}0 & \text{if } n_2>m_2,\\ 1 & \text{if } n_2\leqq m_2.\end{cases}$$

By $C(\chi)$ we mean the Gauss sum $\sum_{n=1}^{q}\chi(n)\exp(-2\pi i n/q)$. We define α_n by $\exp(i\alpha_n)=\chi(n)$, and put $\gamma_n=\alpha_n+(i/4)\log(C(\chi)^2/q)-(\pi/8)$. The symbol $\sum_n'$ indicates the sum in which n runs over the values with $(n,q)=1$. A generalization of Balasubramanian's argument [2] gives

$$\begin{aligned}(2.1)\quad \int_0^T f_1(t)^2dt=&\frac{\varphi(q)}{q}T\log T+\frac{\varphi(q)}{q}\left(\log\frac{q}{2\pi}+2\gamma-1+2\sum_{p|q}\frac{\log p}{p-1}\right)T\\ &-2(2\pi/q)^{1/2}\varphi(q)T^{1/2}+4U-4U_5+2B^*-4B_1-4B_2+4\,\mathrm{Re}(B_3-B_4-B_7)\\ &+4B-4B_{10}+O(q^{7/4}(qT)^{1/4}+q^2(\log q(T+1))^2+q^3\log(q(T+1))),\end{aligned}$$

where

$$U_5=\sum_{m=1}^{L}{}'\sum_{n_2=1}^{q}{}'\sum_{2m_1\leqq n_1\leqq K}\frac{\sin(\alpha_m-\alpha_n+2\pi q n_1^2\log(n/m))}{(mn)^{1/2}\log(n/m)},$$

$$B^*=\left(\frac{2\pi T}{q}\right)^{1/2}\mathrm{Re}\left\{\frac{1}{C(\chi)}\sum_{n_2=1}^{q}{}'\chi(n_2)^2\exp(-2\pi i n_2^2/q)\int_{2q^{-1/2}(q-n_2)}^{\infty}\exp\left(i\pi\left(\theta^2-\frac{1}{4}\right)\right)d\theta\right\},$$

$$B_1=\sum_{m=1}^{L}{}'\sum_{n_2=1}^{q}{}'\sum_{m_1^{1/4}q^{-1/8}<r_1<m_1}\frac{\sin(\alpha_m-\alpha_n+2\pi q n_1^2\log(n/m))}{(mn)^{1/2}\log(n/m)},$$

$$B_2=\mathrm{Im}\left\{\sum_{m=q+1}^{L}{}'\sum_{n_2=1}^{q}{}'\sum_{J\leqq r_1\leqq m_1^{1/4}q^{-1/8}}\frac{\chi(m)\bar{\chi}(n)}{r}(-1)^{r_1}\exp(\pi i(m_2^2-n_2^2)/q)\left(1+\frac{2\pi i r^3}{3mq}\right)\right\},$$

$$B_3=\left(\frac{T}{2\pi q}\right)^{1/2}\sum_{m_2=1}^{q}{}'\sum_{n_2=1}^{q}{}'\sum_{r_1\geqq J}\frac{2\pi\chi(m_2n_2)}{C(\chi)}\exp(-2\pi i m_2n_2/q)$$

$$\times\int_{q^{-1/2}(r+2(q-n_2))}^{\infty}\exp\left(i\pi\left(\theta^2-\frac{1}{4}\right)\right)d\theta,$$

$$B_4=\sum_{m=q+1}^{L}{}' \sum_{n_2=1}^{q}{}' \sum_{r_1>m_1^{1/4}q^{-1/8}} \frac{2\pi\chi(m_2n_2)}{C(\chi)} \exp(-2\pi i m_2 n_2/q)$$
$$\times\int_{q^{1/2}n_1 \log(q^2n_1^2/mn)}^{\infty} \exp\left(i\pi\left(\theta^2-\frac{1}{4}\right)\right)d\theta,$$

$$B_7=\sum_{m=q+1}^{L}{}' \sum_{n_2=1}^{q}{}' \sum_{m_1^{1/4}q^{-1/8}<r_1<m_1} \frac{q^{1/2}\chi(mn)}{iC(\chi)(mn)^{1/2}\log(q^2n_1^2/mn)}$$
$$\times\exp\left(\pi i\left(2qn_1^2\log(q^2n_1^2/mn)-\frac{1}{4}\right)\right),$$

$$B_{10}=\sum_{m=1}^{L}{}' \sum_{n_2=1}^{q}{}' \sum_{2m_1\leq n_1\leq K} \frac{\sin(2\pi qn_1^2\log(q^2n_1^2/mn)+\gamma_m+\gamma_n)}{(mn)^{1/2}\log(q^2n_1^2/mn)},$$

and U and B are as in [4]. The above U_5 and B_j's are analogues of U_5 and B_j's in [2] ($j=1, 2, 3, 4, 7, 10$), while U and B correspond to U_2+U_3 and B_8+B_9 in [2], respectively. We note that in case $q=1$, there appears no term corresponding to the third term in the right-hand side of (2.1).

It is not difficult to see

$$\int_0^T f_2(t)^2dt=2\left(\frac{2\pi T}{q}\right)^{1/2}\int_0^1 G(u)^2du+O(q^2(\log q)^2), \tag{2.2}$$

where

$$G(u)=\frac{1}{\cos(2\pi qu)}\sum_{n=1}^{q}{}' \cos(2\pi(qu+q-2n)u+q^{-1}\pi n^2+\gamma_n).$$

The properties of $G(u)$ are not the same as Balasubramanian's $G(u)$ in [2], so the evaluation of the integral of $f_1(t)f_2(t)$ is rather different from the one in [2]. We have

$$2\int_0^T f_1(t)f_2(t)dt=-4\left(\frac{2\pi T}{q}\right)^{1/2}\int_0^1 G(u)^2du+2\left(\frac{2\pi T}{q}\right)^{1/2}X_0-B_{13}-B_{14}+B_{15} \tag{2.3}$$
$$+O(q^{7/4}(\log q)(qT)^{1/4}+q^{21/8}(\log q)(qT)^{1/8}+q^5(\log q)(\log(T+1))),$$

where

$$X_0=C(\chi)^{1/2}q^{-5/4}\sum_{r=1}^{2q} G\left(\frac{2r-1}{4q}\right)\sum_{n_2=1}^{q}{}' \bar{\chi}(n_2)$$
$$\times\exp\left\{-\frac{\pi i}{8q}(2r-1)^2+\frac{\pi i}{q}n_2(2r-1)-\frac{\pi i}{q}n_2^2+\frac{\pi i}{8}\right\},$$

$$B_{13}=8G(0)q^{-1/2}\sum_{n=1}^{L}{}' n^{-1/2}\sum_{n_1+n_1^{1/4}q^{-1/8}<d\leq 2(n_1-1)} d^{-1/2}(\log(qd/n))^{-1}$$
$$\times\sin(2\pi qd^2\log(qd/n)+\gamma_n),$$

$$B_{14}=8G(0)q^{-1/2}\sum_{n=1}^{L}{}' n^{-1/2}\sum_{2(n_1-1)<d\leq K}\frac{\sin(2\pi qd^2\log(qd/n)+\gamma_n)}{d^{1/2}\log(qd/n)}$$

and

$$B_{15}=4G(0)q^{-1/2}\sum_{n=1}^{L}{}' \frac{\sin(2\pi qd_0^2\log(qd_0/n)+\gamma_n)}{(nd_0)^{1/2}\log(qd_0/n)}$$

(d_0 is the smallest integer satisfying $d_0>n_1+n_1^{1/4}q^{-1/8}$). The appearance of the term including X_0 is the most different feature from the case of the Riemann zeta-function $\zeta(s)$. This term comes from certain residues connected with the function $G(u)$, and thanks to the existence of this term, the third term in the right-hand side of (2.1) is finally cancelled.

From (2.1), (2.2), (2.3) and Schwarz' inequality, we see that the other integrals in the right-hand side of (1.3) are included in $E(T,\chi)$. Moreover, we can prove

$$-4B_1-4\,\mathrm{Re}\,B_7-B_{13}=O(q^{11/8}(qT)^{1/4}+q^2(\log(T+1))^2+q^{9/2}),$$
$$-4U_5-4B_{10}-B_{14}=O(q^{3/2}(qT)^{1/4}+q^2\log(T+1)+q^2\log q),$$

and, by using the fact

$$\mathrm{Im}\left\{\sum_{m_2=1}^{q}{}'\sum_{n_2=1}^{q}{}'\chi(m_2)\bar{\chi}(n_2)\exp(\pi i(m_2^2-n_2^2)/q)\sum_{r_1\geqq J}(-1)^{r_1}r^{-1}\right\}=0,$$

we can also prove

$$-4B_2-4\,\mathrm{Re}\,B_4+B_{15}=O(q^{5/4}(\log q)(qT)^{1/4}+q^{9/2}).$$

Next, by the method indicated in [4], we can show that U and B are included in $E(T,\chi)$; here we use Balasubramanian's multiple integration process and Weil's estimate on Kloosterman sums.

§ 3. Now the only task remaining to us is to prove

$$(3.1)\qquad -2(2\pi/q)^{1/2}\varphi(q)T^{1/2}+2B^*+4\,\mathrm{Re}\,B_3-2(2\pi T/q)^{1/2}\int_0^1 G(u)^2du+2(2\pi T/q)^{1/2}X_0=0.$$

In the case of $\zeta(s)$, two proofs of the corresponding fact are given in the last page of Titchmarsh [11]. Both proofs can be generalized to our case. A generalization of the first (indirect) proof can be obtained by using the formula stated in the last page of Kober [3]. The second way is a direct one, and a generalization of Titchmarsh's argument gives that the left-hand side of (3.1) is equal to

$$(3.2)\qquad 2(2\pi T/q)^{1/2}\left(-\varphi(q)+\pi i\sum_{r=1}^{2q}P_r+X_0\right),$$

where P_r is the residue of the function

$$\frac{1}{2\cos^2(2\pi qu)}\sum_{m=1}^{q}{}'\sum_{n=1}^{q}{}'\exp\left\{-i(4\pi qu^2+4\pi(q-m-n)u+\frac{\pi}{q}(m^2+n^2)+\gamma_m+\gamma_n)\right\}$$

at $u=(2r-1)/4q$. Finally, by using the fact

$$\sum_{m=1}^{q}{}'\sum_{\substack{n=1\\ m\neq n}}^{q}{}'\chi(m)\bar{\chi}(n)\exp(\pi i(m^2-n^2)/q)\frac{\exp(-\pi i(m-n)/q)}{1-\exp(-2\pi i(m-n)/q)}=0,$$

we can show the quantity (3.2) is equal to zero. This implies the assertion of the theorem.

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