54. A Generalization of the Chowla-Selberg Formula and the Zeta Functions of Quadratic Orders

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The purpose of this note is to present an identity which generalizes a formula of Chowla and Selberg on the periods of CM elliptic curves in connection with the zeta functions of imaginary quadratic orders. We first proposed the identity as the numerical evidence and then Y. Nakkajima and Y. Taguchi have proved it algebraically by using the technique from arithmetic geometry ([2]). We employ here an analytical approach using zeta functions.

1. Review and result. Let K be an imaginary quadratic field with discriminant D, O_K its ring of integers, w the order of unit group $O_K^\times$, and χ the quadratic Dirichlet character modulo $|D|$ determined by the extension $K/\boldsymbol{Q}$. For a lattice L in $\boldsymbol{C}$, define

$$g_2(L)=60\sum_{\lambda\in L}{}' \lambda^{-4},\ g_3(L)=140\sum_{\lambda\in L}{}' \lambda^{-6}$$

and

$$\Delta(L)=g_2(L)^3-27g_3(L)^2.$$

The "discriminant" $\Delta(L)$ is non-zero and has the property

$$\Delta(\alpha L)=\alpha^{-12}\Delta(L) \qquad \textit{for } \alpha\in \boldsymbol{C}^\times. \tag{1}$$

Take an ideal $\mathfrak{A}$ of K and consider the value

$$F(\mathfrak{A})=\Delta(\mathfrak{A})\Delta(\mathfrak{A}^{-1}).$$

By (1) it depends only on the class of $\mathfrak{A}$ in the ideal class group $Cl(O_K)$. Any period of elliptic curves with complex multiplication in K differs by an algebraic constant from the 24-th root of $F(\mathfrak{A})$. The formula of Chowla-Selberg expresses the product of $F(\mathfrak{A})$ over $Cl(O_K)$ by gamma values:

$$\prod_{Cl(O_K)} F(\mathfrak{A})=\left(\frac{2\pi}{|D|}\right)^{12h}\prod_{a=1}^{|D|-1}\Gamma\left(\frac{a}{|D|}\right)^{6w\chi(a)}. \tag{2}$$

Now let O_f be an order of K of conductor f. We denote by $Cl(O_f)$ the group of proper O_f-ideal classes and by h_f its order (the class number of O_f). The function $F(\mathfrak{A})$ is defined also on the proper O_f-ideals $\mathfrak{A}$ and again depends only on the class in $Cl(O_f)$. Y. Nakkajima and Y. Taguchi ([2]), inspired by K. Fujiwara, gave an algebraic proof (assuming (2), though) of the following

Theorem. *We have*

$$\prod_{Cl(O_f)} F(\mathfrak{A})=\left(\frac{2\pi}{|f^2D|}\right)^{12h_f}\left(\prod_{a=1}^{|D|-1}\Gamma\left(\frac{a}{|D|}\right)^{6w\chi(a)}\right)^{h_f/h}\times\prod_{p|f}p^{12e(p)}, \tag{3}$$

where

$$e(p)=(1-\chi(p))\frac{p^n-1}{p-1}\frac{h_f}{p^n\left(1-\frac{\chi(p)}{p}\right)}=\frac{(1-p^{-n})(1-\chi(p))}{(1-p^{-1})(p-\chi(p))}\cdot h_f$$

with p^n the highest power of p dividing f.

They used the interpretation of the left hand sides of (2) and (3) as the heights (in the sense of Faltings) of elliptic curves over number fields. From this point of view, the identity (3) supplies an example of explicit description of the behavior of heights under isogeny. Another proof described below is an analytic one similar to that of the original identity (2) (cf. [1], [3]), namely, it uses Kronecker's limit formula and a formula relating $L'(0,\chi)$ with gamma function.

2. Outline of proof. Define the zeta function of O_f by

$$\zeta_{O_f}(s)=\sum_{\mathfrak{A}:proper}\frac{1}{N(\mathfrak{A})^s}.$$

Here, $\mathfrak{A}$ runs over all proper (integral) O_f-ideals and $N(\mathfrak{A})=|O_f/\mathfrak{A}|$. The series converges absolutely and uniformly in the half plane $Re(s)>1$. The following proposition gives the Euler product of $\zeta_{O_f}(s)$, which is based on the factorization

$$(3)\qquad \zeta_{O_f}(s)=\prod_{p:prime}\sum_{\substack{\mathfrak{A}^{(p)}:proper\\ N(\mathfrak{A}^{(p)})=p-power}}\frac{1}{N(\mathfrak{A}^{(p)})^s}.$$

We note here that the factorization of a proper ideal into "irreducible" proper ideals is *not* necessarily unique but the factorization into proper ideals with mutually coprime prime-power norms is unique, hence the factorization (3) holds.

Proposition. *We have*

$$(4)\qquad \zeta_{O_f}(s)=\prod_{p|f}\frac{(1-p^{-s})(1-\chi(p)p^{-s})-p^{n-1-2ns}(1-p^{1-s})(\chi(p)-p^{1-s})}{1-p^{1-2s}}\times\zeta_K(s),$$

where p^n is the highest power of p dividing f and $\zeta_K(s)$ is the Dedekind zeta function of K.

If we put $p^{-s}=u$, the factor

$$\begin{aligned}f(u)&=\frac{(1-p^{-s})(1-\chi(p)p^{-s})-p^{n-1-2ns}(1-p^{1-s})(\chi(p)-p^{1-s})}{1-p^{1-2s}}\\&=\frac{(1-u)(1-\chi(p)u)-p^{n-1}u^{2n}(1-pu)(\chi(p)-pu)}{1-pu^2}\end{aligned}$$

turns out to be a polynomial in u of degree $2n$ and satisfies a functional equation

$$f(u)=p^nu^{2n}f\left(\frac{1}{pu}\right).$$

Moreover, we see that every root of $f(u)=0$ has its absolute value $p^{-1/2}$. Hence, as a corollary of the proposition, the function $\zeta_{O_f}(s)$ has meromorphic continuation to the whole s-plane, satisfies a functional equation

$$(2\pi)^{1-s}\Gamma(s)\zeta_{O_f}(s)=|f^2D|^{\frac{1}{2}-s}(2\pi)^s\Gamma(1-s)\zeta_{O_f}(1-s),$$

and the Riemann hypothesis for $\zeta_{O_f}(s)$ submit to that for $\zeta_K(s)$.

We can deduce Theorem from Proposition as in Weil [3, Chapter IX]. First we obtain from Kronecker's first limit formula

$$\zeta'_{O_f}(0)=-\frac{1}{12w_f}\sum_{Cl(O_f)}\log F(\mathfrak{A}) \qquad (w_f=|O_f^{\times}|).$$

This together with the calculation of the derivative at $s=0$ of the right hand side of (4), using the identity $\zeta_K(s)=\zeta(s)L(s,\chi)$ ($\zeta(s)$: Riemann zeta function, $L(s,\chi)$: Dirichlet L-function attached to χ), yields the theorem.

References

[1] S. Chowla and A. Selberg: On Epstein's zeta-function. Crelle J., **227**, 86–110 (1967).

[2] Y. Nakkajima: On the periods of elliptic curves with complex multiplication by an imaginary quadratic order—A generalization of the Chowla-Selberg formula (in Japanese, with an appendix by Y. Taguchi). Master's thesis, The University of Tokyo (1989).

[3] A. Weil: Elliptic Functions According to Eisenstein and Kronecker. Springer-Verlag (1976).