49. Note on Isometry Invariant Geodesic on Two Dimensional Spherical Manifold*),**)

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Let f be an isometry on a Riemannian manifold. Then a geodesic c is called "f-invariant" (or isometry invariant geodesic) if $f(c(t))=c(t+1)$ for any $t \in R$. In general such a geodesic is not necessary closed except isometry of finite order. (We studied the structure of such a geodesic for an isometry of finite order [2], [3].) It is very interesting to ask what isometry has an invariant closed geodesic. There is no general information about this problem other than the following theorem: *if there is only a finite number of geodesics, then it is closed* [1] (cf. [5]). In this note we assert that any isometry of small displacement on two dimensional spherical manifold has an invariant closed geodesic. Though our result can be proved by using Theorem 3.2 [1], we give here directly another proof because our method is very elementary and more geometrical.

An isometry f on a Riemannian manifold M is called "small displacement" if for each $x \in M$ there is a unique minimizing geodesic from x to $f(x)$. Our main result is the following

Theorem. *Let M be a Riemannian manifold homeomorphic to S^2. Let f be a small displacement. Then there is a closed f-invariant geodesic which is not a point curve.*

The real valued function δ_f on M is defined by $\delta_f(x)=d(x, f(x))$ where d is a distance function of M and f is an isometry. In [4] Ozols has studied the critical point of δ_f. Let $\mathrm{Crt}(f)$ be a set of critical point of δ_f^2, then $\mathrm{Crt}(f)=F(f) \cup$ (critical point of δ_f on $(M-F(f))$ where $F(f)$ is a set of a fixed point of f.

Fact. *Let f be a small displacement. Then $x \in Crt(f)-F(f)$ if and only if f preserves the minimizing geodesic from x to $f(x)$.*

If M is compact, then δ_f^2 has a maximum point on M. Thus we have

Lemma 1. *Let f be a small displacement on a compact manifold. Then there exists a f-invariant geodesic which is not a point curve.*

By this lemma we have only to prove the theorem when our isometry is not finite order. From now on we assume that Riemannian manifold is homeomorphic to S^2 and f is a small displacement of which order is not finite. Since f is a small displacement, f is homeomorphic to 1 and so

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$\# F(f)=2$.

Lemma 2. *If $F(f)=\{p, q\}$, then q is a cut point of p.*

Proof. Let $T_p(M)$ be a tangent plane of M at p. Then the differential f_* at p is a rotation on $T_p(M)$. The rotation is irrational because the order of f is not finite. Now let $v_p \in T_p(M)$ be a vector such that $\exp_p(v_p)=q$ and $|v_p|=r$. Since $\{f_*^n(v_p)\}$ $(n \in Z)$ is dense in $S_p(r)=\{u \in T_p(M); |u|=r\}$, there exists a subsequence $\{f_*^{n_i}(v_p)\}$ such that $\lim_{i\to\infty} f_*^{n_i}(v_p)=u$ for any $u \in S_p(r)$. Thus we have $\exp_p(u)=\exp_p(\lim_{i\to\infty} f_*^{n_i}(v_p))=\lim_{i\to\infty} f^{n_i}(\exp_p(v_p))=q$ which implies q is a cut point of p. Q.E.D.

The proof of Lemma 2 implies that the following lemma which means M is just like a surface of revolution.

Lemma 3. *Let C_{pq} be a minimal geodesic from p to q. Then we have $M=\overline{\bigcup_{n\in Z} f^n(C_{pq})}$.*

Proof. Let v_p be a tangent vector of M at p such that $C_{pq}(t)=\exp_p(tv_p)$ $(0 \leqq t \leqq 1)$ and $|v_p|=r$. Let x be any point of M which x is not a cut point of p, we can take a subsequence $\{f_*^{n_i}((a/r)v_p)\}$ such that $\lim_{i\to\infty} f_*^{n_i}((a/r)v_p)=u$ for $u=\exp_p^{-1}(x)$ because $\{f_*^n((a/r)v_p)\}$ is dense in $S_p(a)$. Since $\exp_p((a/r)v_p) \in C_{pq}$ and $\lim_{i\to\infty} f^{n_i}(\exp_p((a/r)v_p)=x$, we have $x \in \overline{\bigcup_{n\in Z} f^n(C_{pq})}$. On the other hand let x is a cut point of p, then $x=q$ and so consequently we have $M=\overline{\bigcup_{n\in Z} f^n(C_{pq})}$.

Lemma 4. *Let C_{pq} be a minimizing geodesic from p to q. If $\delta_f(x_0)=\max_{x\in C_{pq}} \delta_f(x)$, then the minimizing geodesic from x_0 to $f(x_0)$ is f-invariant.*

Proof. We have only to show that δ_f is local maximum around x_0. Then the conclusion follows from Fact. Let y be a point near x_0. Then there is a point $x \in C_{pq}$ by Lemma 3 such that $\lim_{i\to\infty} f^{n_i}(x)=y$ and the point x is also near x_0. Since x_0 is a maximum point of δ_f on C_{pq}, $d(x_0, f(x_0)) \geqq d(x, f(x))=d(f^{n_i}(x), f(f^{n_i}(x)))$ and so $d(x_0, f(x_0)) \geqq \lim_{i\to\infty} d(f^{n_i}(x), f(f^{n_i}(x)))$ $=d(y, f(y))$ which implies δ_f is local maximum at x_0. Q.E.D.

Proof of Theorem. Let c be a geodesic obtained by Lemma 4 with $c(0) \in C_{pq}$ and $f(c(0))=c(r) \in f(C_{pq})$. Here we have only to show c is a closed curve. Let $t_1=\min\{t \mid d(p, c(t))=k_1, 0 \leqq t \leqq 1\}$ and $t_2=\max\{t \mid d(p, c(t))=k_2, 0 \leqq t \leqq 1\}$ where $k_1=\min\{d(p, x) \mid x \in c([0, r])\}$ and $k_2=\max\{d(p, x) \mid x \in c([0, r])\}$. Without loss of generality we can assume that $t_1 \leqq t_2$.

Case I. $t_1=t_2$: This implies $k_1=k_2$. On the other hand by Lemma 3 we have $c=\bigcup f^n(c([0, r]))$ because c is f-invariant. Thus we have c is closed.

Case II. $t_1<t_2$: If $k_1=k_2$, c is closed by the same argument as the case I. And so assume $k_1 \neq k_2$. (1) Suppose $c(t_1) \notin C_{pq}$. Let x_i $(i=1, 2)$ be points on C_{pq} and $f(C_{pq})$ respectively with $d(p, x_i)=d(p, c(t_i))$. Put $\varepsilon=\min\{d(x_1, c(t_1)), d(x_2, c(t_2))\}$, then there is n such that $\max_{x\in c([0,r])} d(x, f^n(x))<\varepsilon/5$ by the above $c=\bigcup f^n(c([0, r]))$ and by Lemma 3. Here we can assume that $d(x_1, c(t_1))<d(x_1, f^n(c(t_1))$ and $d(x_2, c(t_2))>d(x_2, f^n(c(t_2))$ without loss of generality. Since $d(p, c(t_i))=d(p, f^n(c(t_i))$ $(i=1, 2)$ and $\dim M=2$, the geodesic $c\mid[0, r]$ and $f^n \cdot c\mid[0, r]$ intersect at least two points y_1, y_2. Thus y_1, y_2 are joined by two geodesic arcs: one is a part of $c|[0,r]$ and the other is a part of $f^n\cdot c|[0,r]$. However this is impossible because $c|[0,r]$ is a unique minimal geodesic joining $c(0)$ and $c(r)$. (2) Suppose $c(t_1)\in C_{pq}$ (i.e. $c(t_1)=c(0)$), then $c(t_2)$ must be an interior point of $c([0,r])$ because $k_1\neq k_2$. Put $\varepsilon= d(x_2, c(t_2))$ then we have a contradiction as same as (1). Thus (1) and (2) imply $k_1=k_2$ and hence c is closed. The both cases imply c is a closed curve which is not a point curve by our construction. Q.E.D.

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