48. *Twisting Symmetry-spins of Pretzel Knots**)

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Let π be the commutator subgroup of the knot group of a knot in the 4-sphere S^4. In [1] it is shown that if π is finite, then $\pi = G \times Z_d$ where $G = \{1\}$, the quaternion group $Q(8)$, the binary icosahedral group I^* or the generalized binary tetrahedral group $T(k)$ and d is an odd integer which is relatively prime to the order of G. Conversely, Yoshikawa [10] has shown that these groups can be realized as the commutator subgroups of the knot groups of knots in S^4 except $Q(8) \times Z_d$, $d > 1$. Actually, these knots were constructed by twist-spinning certain 2-bridge knots and pretzel knots. The exceptional groups were realized only as the commutator subgroups of knot groups of knots in homotopy 4-spheres. Note that $Q(8) \times Z_d$ is isomorphic to the fundamental group of a prism manifold M_d, that is, the Seifert fibered manifold with invariants $\{b : (o_1, 0) : (2,1), (2,1), (2,1)\}$, $d = |2b+3|$ (cf. [3], [7]). Since then, by using the deform-spinning introduced by Litherland [6], Kanenobu [4] and the author [9] showed that for $d = 5$, 11, 13 and 19 (equivalently $b = -4, 4, -8$ and 8), there is a fibered 2-knot in S^4 whose fiber is the punctured prism manifold M_d°; thus for these values of d, the groups $Q(8) \times Z_d$ are realized as the commutator subgroups of knot groups of knots in S^4. It should be noted that a fibered 2-knot with fiber $M_d^\circ (d > 1)$ cannot be constructed by twist-spins (cf. [2]).

The purpose of this paper is to show that other three values can be realized.

Theorem. *There exists a fibered 2-knot in S^4 whose fiber is a punctured prism manifold M_d° with fundamental group isomorphic to $Q(8) \times Z_d$ for $d =$ 3, 5, 11, 13, 19, 21, 27.*

Our examples for the cases $d = 3$, 21, 27 will be constructed by a product of two symmetry-spinnings and 1-twist-spinning for pretzel knots. It is unknown whether there exists such a fibered 2-knot in S^4 for any other value of d.

All maps and spaces are assumed to be in the PL category, and all manifolds are oriented. A circle is identified with the quotient space $\boldsymbol{R}/\boldsymbol{Z}$. The unit interval $[0, 1]$ is denoted by I.

1. Construction. Let (S^3, K) be a knot and suppose that there are orientation-preserving periodic homeomorphisms $g_i (i = 1, 2)$ on (S^3, K) of order n_i such that $g_1 g_2 = g_2 g_1$, $(n_1, n_2) = 1$, and $J_1 \cup J_2$ is the Hopf link with $lk(J_1, J_2) = 1$, where $J_i = \mathrm{Fix}(g_i)$, $(i = 1, 2)$. Let $n = n_1 n_2$, $g = g_1 g_2$. Let q:

*) Dedicated to Professor Junzo TAO on his 60th birthday.

$S^3 \to S^3/g$ be the quotient map, and $\bar{K}=q(K)$, $\bar{J}_i=q(J_i)$. The map q is the $Z_{n_1}\oplus Z_{n_2}$-branched cover branched over $\bar{J}_1\cup\bar{J}_2$, corresponding to Ker $[\pi_1(S^3-\bar{J}_1\cup\bar{J}_2)\to H_1(S^3-\bar{J}_1\cup\bar{J}_2)\to Z_{n_1}\oplus Z_{n_2}]$, where the last homomorphism sends a meridian $t_1(t_2$ resp.) of $\bar{J}_1$ ($\bar{J}_2$ resp.) to $(1,0)$ ($(0,1)$ resp.) $\in Z_{n_1}\oplus Z_{n_2}$. Let $\bar{K}\times D^2$ be a tubular neighbourhood of $\bar{K}$ disjoint from $\bar{J}_1$ and $\bar{J}_2$, and $X(\bar{K})=cl(S^3-\bar{K}\times D^2)$. It is well-known that there is a map $\bar{p}: X(\bar{K})\to\partial D^2$ such that $\bar{p}|\partial X(\bar{K}): \partial X(\bar{K})=\bar{K}\times\partial D^2\to\partial D^2$ is the projection (cf. [5: Ch. 3], [8: Ch. 5]). Then $q^{-1}(\bar{K}\times D^2)$ is a g-invariant tubular neighbourhood $K\times D^2$ of K with $q(x,v)=(nx,v)$, $x\in K$, $v\in D^2$. We always assume that $K\times v(v\in\partial D^2)$ is null-homologous in $X(K)=cl(S^3-K\times D^2)$. Since $j_i=lk(K,J_i)$ is coprime to n_i, we can choose an integer k_i such that $j_ik_i\equiv 1 \pmod{n_i}$. It follows that $g_i|K\times D^2$ is given by $(x,v)\to(x+k_i/n_i,v)$. Then $g|K\times D^2$: $(x,v)\to(x+k/n,v)$, $k=k_2n_1+k_1n_2$. Take a collar $\partial X(K)\times I$ of $\partial X(K)=K\times\partial D^2$ such that $\partial X(K)$ is identified with $\partial X(K)\times\{0\}$, which is disjoint from J_1 and J_2. Define two homeomorphisms $t, s_{n,k}: (S^3,K)\to(S^3,K)$ as follows:

$$\begin{aligned}
t(x,\theta,\phi)&=(x,\theta+\phi,\phi) && \text{for } (x,\theta,\phi)\in K\times\partial D^2\times I,\\
t(y)&=y && \text{for } y\notin\partial X(K)\times I,\\
s_{n,k}(x,\theta,\phi)&=(x-k(1-\phi)/n,\theta,\phi) && \text{for } (x,\theta,\phi)\in K\times\partial D^2\times I,\\
s_{n,k}(x,v)&=(x-k/n,v) && \text{for } (x,v)\in K\times D^2,\\
s_{n,k}(y)&=y && \text{for } y\in X(K)-\partial X(K)\times I.
\end{aligned}$$

Then $s_{n,k}g|K\times D^2=id$, $s_{n,k}g|cl(X(K)-\partial X(K)\times I)=g$, and $\bar{p}q(s_{n,k}g|X(K))=\bar{p}q$. Note that $\bar{p}q: X(K)\to\partial D^2$ is the map whose restriction $\bar{p}q|\partial X(K)$: $\partial X(K)=K\times\partial D^2\to\partial D^2$ is the projection. Fix a point x on K. Take a ball neighbourhood of K_- of x in K, and set $B_-=K_-\times D^2$. Then (B_-,K_-) is a standard ball pair. Let (B_+,K_+) be the complementary ball pair. For any nonzero integer m, construct $\partial(B_+,K_+)\times B^2\cup_\partial(B_+,K_+)\times_{t^m s_{n,k}g}\partial B^2$. This is a locally flat sphere pair depending only on the isotopy classes τ of t, and $\omega_{n,k}$ of $s_{n,k}g$ (rel $K\times D^2$) [6: Lemma 1.2]. We write $\tau^m\omega_{n,k}K$ for this 2-knot in S^4. Remark that $\omega_{n,k}$ is an untwisted deformation with respect to $(\bar{p}q, K\times D^2)$ in terms of [6]. The main theorem of [6] states that $\tau^m\omega_{n,k}K$ is fibered.

2. The fiber. Let a,b be coprime integers with $b\neq 0$. Let $\Phi: K\times\partial D^2\to K\times\partial D^2$ be a homeomorphism $(x,\theta)\to(x+b\theta,a\theta)$. By $S^3(K,a/b)$ we mean the manifold obtained from S^3 by removing $K\times D^2$ and sewing it back using Φ. Let K^* denote the image of $K\times\{0\}$ under this surgery. Moreover, for any integers c,d with $d\neq 0$, choose coprime integers a,b with $a/b=c/d$, and let $S^3(K,c/d)=S^3(K,a/b)$.

Proposition. *Let (S^2,K) be a knot having the property as described in Section 1. Let $\bar{K},\bar{J}_i,k_i$ $(i=1,2)$, $k=k_2n_1+k_1n_2$, n be as before. For $m>0$, let M be the mn-fold cyclic branched covering space of $S^3(\bar{K},m/k)$ branched over $\bar{K}^*\cup\bar{J}_1\cup\bar{J}_2$, corresponding to* Ker $[\pi_1(S^3-\bar{K}\cup\bar{J}_1\cup\bar{J}_2)\to Z\langle t_0\rangle\times Z\langle t_1\rangle\times Z\langle t_2\rangle\to Z_{mn}\langle t\rangle]$. *Here $t_0(t_1,t_2$* resp.) *corresponds to a meridian of $\bar{K}(\bar{J}_1,\bar{J}_2$* resp.) *and the last homomorphism sends t_0 to t, and t_1t_2 to t^{-m}. Then the fiber of $\tau^m\omega_{n,k}K$ is M°.*

Note that the projection $M \to S^3(\bar{K}, m/k)$ is n to 1 over $\bar{K}^*$, mn_2 to 1 over $\bar{J}_1$, mn_1 to 1 over $\bar{J}_2$. This proposition is a generalization of Proposition 5.4 of [6], and can be proved similarly. We shall show its sketch and how to identify the manifold M.

Sketch of the proof. In [6] it is shown that the closed fiber is $M = K \times D^2 \cup_\beta \{(y, \phi) \in X(K) \times_{s_{n,kg}} S^1 \mid p(y) = m\phi\}$, where β: $K \times \partial B^2 \to \{(y, \phi) \in \partial X(K) \times_{s_{n,kg}} S^1 \mid p(y) = m\phi\}$ is given by $(x, \phi) \to ((x, m\phi), \phi)$, and $p = \bar{p}q$. Then g acts on M naturally, since $pg = p$. Let $M_1 = M/g$. It is easy to see that M_1 is obtained from $\Sigma_m(\bar{K})$, the m-fold cyclic branched covering space of S^3 over $\bar{K}$, by performing $1/k$-surgery (with respect to the induced framing) along the lift of $\bar{K}$. Thus M_1 is the m-fold cyclic branched covering space of $S^3(\bar{K}, m/k)$ over $\bar{K}^*$. These observations imply that M is as described in Proposition.

Given such a knot K, we can construct M as follows. Take $\Sigma_m(\bar{K})$ and let $\tilde{J}_i$ be the lift of $\bar{J}_i$ $(i=1, 2)$, which is not necessarily connected. Let M_1 be the manifold obtained from $\Sigma_m(\bar{K})$ by performing $1/k$-surgery along the lift of $\bar{K}$, and let $\tilde{J}_i^*$ be the image of $\tilde{J}_i$. Finally take the $Z_{n_1} \oplus Z_{n_2}$-branched covering space of M_1 over $\tilde{J}_1^* \cup \tilde{J}_2^*$, and we get M. In particular, if $\bar{K}$ is unknotted, then $\Sigma_m(\bar{K})$ and M_1 are homeomorphic to S^3. Actually we will deal with only this case.

3. Proof of Theorem. Let $P(m, n)$ be the pretzel knot as illustrated in Fig. 1, where n is an odd integer, and $2m+1$ denotes the number of half-twists (left-handed if $m \geq 0$, right if $m < 0$). Note that $P(0, n)$ and $P(-1, n)$ are torus knots of type $(2, n)$, $(2, -n)$, respectively. It is clear that $P(m, n)$ has two symmetries g_1 of order n, and g_2 of order 2 such that $g_1 g_2 = g_2 g_1$. Put $J_i = \mathrm{Fix}(g_i)$ $(i=1, 2)$, and orient them such that $lk(P(m, n), J_1) = 2$, $lk(P(m, n), J_2) = (-1)^m n$, $lk(J_1, J_2) = 1$. Thus the knot $P(m, n)$ has the property as described in Section 1. By considering a suitable power of g_1, we may assume $k = \pm 1$, and consider these cases.

Lemma 1. *Let $P(m, n)$ be as above. Then the closed fiber of $\tau^1\omega_{2n,k}P(m, n)$, $k = \pm 1$, is given as follows:*

(1) *the Seifert fibered manifold* $\{0: (o_1, 0): (m, 1), \cdots n \cdots, (m, 1)\}$ *if* $k=1$ *and* $m \neq 0$,

(2) *the Seifert fibered manifold* $\{0: (o_1, 0): (m+1, 1), \cdots n \cdots, (m+1, 1)\}$, *if* $k = -1$ *and* $m \neq -1$,

(3) $\#^{n-1} S^2 \times S^1$, *if* $k=1$ *and* $m=0$, *or* $k=-1$ *and* $m=-1$.

Proof. We shall follow the procedure given in Section 2 in determining the closed fiber. Let $q: S^3 \to S^3/g_1 g_2$ be the quotient map, let $\bar{P}(m, n) = q(P(m, n))$, $\bar{J}_i = q(J_i)$ $(i=1, 2)$. Note that $\bar{P}(m, n)$ is unknotted (Fig. 1). Since we consider the 1-twist-spinning, M_1 is obtained from S^3 by performing $1/k$-surgery along $\bar{P}(m, n)$, and it follows that M_1 is homeomorphic to S^3. Trivialize the surgery by $(-k)$-twist (cf. [8]), and let J_i^* be the image of $\bar{J}_i$ under $(-k)$-twist $(i=1, 2)$ (Fig. 2). Finally we must take the $Z_n \oplus Z_2$-branched covering space of M_1 over $J_1^* \cup J_2^*$, corresponding to Ker $[\pi_1(M_1 -$

$J_1^* \cup J_2^*) \to Z\langle t_1\rangle \times Z\langle t_2\rangle \to Z_n \oplus Z_2]$, where the last homomorphism sends a meridian t_1(t_2 resp.) of J_1^*(J_2^* resp.) to (1, 0) ((0, 1) resp.). Take the n-fold cyclic branched covering over J_1^*, and identify the lift $\tilde{J}_2^*$ of J_2^*. The result follows by taking the 2-fold branched covering over $\tilde{J}_2^*$.

Let $Q(m, n)$ be the pretzel knot as illustrated in Fig. 3, where n is an odd integer, $2m+1$ denotes the number of half-twists (left-handed if $m \geq 0$, right if $m<0$). It is clear that $Q(m, n)$ has two symmetries g_1 of order n, and g_2 of order 2, and has the property as described in Section 1. We may assume $k=1$, and consider this case.

Lemma 2. *Let $Q(m, n)$ be as above. Then the closed fiber of $\tau^1\omega_{2n,1}$ $Q(m, n)$ is given as follows:*

(1) *the Seifert fibered manifold* $\{-4n: (o_1, 0): (m+1, 1), \cdots n \cdots, (m+1, 1)\}$, *if* $m \neq -1$,

(2) $\#^{n-1} S^2 \times S^1$, *if* $m=-1$.

Proof. We can determine the closed fiber in the same way as the proof of Lemma 1. See Fig. 4.

Proof of Theorem. In Lemma 1(1) take $(m, n)=(2, 3)$, or in Lemma 1(2) take $(m, n)=(1, 3)$. Then in either case we get the prism manifold M_3. In Lemma 2(1) take $(m, n)=(1, 3)$, $(-3, 3)$. Then we get the prism manifolds M_{21}, M_{27}, respectively.

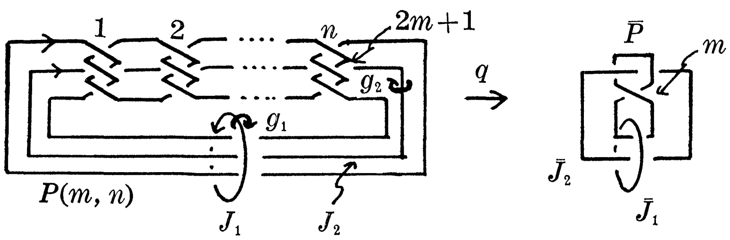

Fig. 1

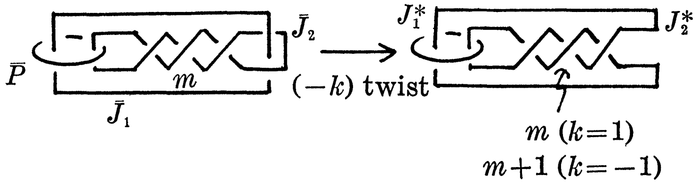

Fig. 2

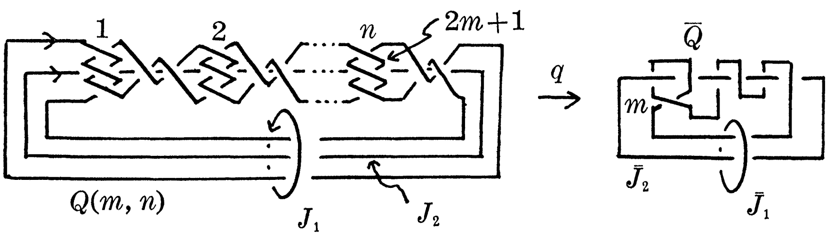

Fig. 3

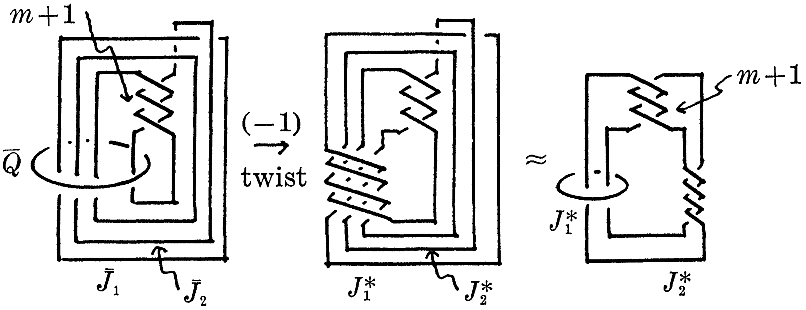

Fig. 4

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