41. On a Generalization of MacPherson's Chern Homology Class. II

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§0. **Introduction.** For a non-singular variety X, a characteristic class of X is defined to be that of its tangent bundle TX and any characteristic class of X is expressed as a polynomial of individual Chern classes of X, in which sense Chern classes are fundamental characteristic classes for the case of non-singular varieties. As for the case of singular varieties, there is at the moment no general notion available of characteristic classes, mainly because one cannot define the tangent bundle unlike in the smooth case, although there are some notions of tangents, such as "tangent cone" [7] and "tangent star" [3]. For the Chern class and the Todd class, there are singular versions; namely, Deligne-Grothendieck-MacPherson's theory C_* (abbr. *DGM-theory*) of Chern class [4] and Baum-Fulton-MacPherson's theory Td_* (abbr. *BFM-theory*) of Todd class [1] (also see [5]). They are both formulated as unique natural transformations from certain group (covariant) functors to the homology group (covariant) functor such that they satisfy certain "smooth condition" (see below). Motivated by MacPherson's survey article [5] and the formulations of DGM-theory C_* and BFM-theory Td_*, the author [8] extended DGM-theory C_* of (total) Chern class to a "DGM-type" theory C_{t*} of the Chern polynomial, which includes DGM-theory C_* as a special case. In this note we give a characterization of "DGM-type" theories of characteristic classes under certain conditions.

§1. **DGM-theory of Chern class and pushforward stability.** Let $\mathcal{CV}$ be the category of compact complex algebraic varieties and $\mathcal{Ab}$ be the category of abelian groups. Let $\mathcal{F}: \mathcal{CV} \to \mathcal{Ab}$ be the correspondence such that for any $X \in \mathrm{Obj}(\mathcal{CV})$ $\mathcal{F}(X)$ is defined to be the abelian group of constructible functions on X. If we define the pushforward $f_* := \mathcal{F}(f)$ for $f: X \to Y$ by $f_*(1_W)(y) := \chi(f^{-1}(y) \cap W)$, then the correspondence $\mathcal{F}$ becomes a covariant functor with this "topologically defined" pushforward [4]. Let $H_*(\ ; Z)$ be the usual Z-homology group (covariant) functor. Then Deligne and Grothendieck conjectured and MacPherson proved the following, using Chern-Mather classes and his graph construction method:

Theorem 1.1 (*DGM-theory C_* of Chern class,* [4]). *There exists a unique natural transformation $C_*: \mathcal{F} \to H_*(\ ; Z)$ satisfying "smooth condition" that $C_*(1_X) = c(TX) \cap [X]$ for any smooth variety X, where 1_X is the*

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characteristic function on X and $c(TX)$ is the usual total cohomology Chern class of the tangent bundle TX.

It is not hard to show by resolution of singularities that "smooth condition" implies the uniqueness of such a natural transformation. For any variety X $\mathcal{F}(X)$ is generated by characteristic functions 1_W's, where W runs through all subvarieties of X. However, there is another set of generators; namely, $\mathcal{F}(X)$ is freely generated by local Euler obstructions, which are constructible (see [2, 4]), i.e., $\mathcal{F}(X)=\{\sum_W n_W Eu_W \mid W$ runs through all subvarieties of X, $n_W \in Z$ and Eu_W is MacPherson's local Euler obstruction$\}$. Then, what is crucial and fundamental in MacPherson's proof is: For any $f: X \to Y$ and any subvariety $W \subset X$, there exist distinguished subvarieties V_i's in $f(W) \subset Y$ and integers n_i's such that (A) $f_*\hat{C}(W)=\sum_i n_i \hat{C}(V_i)$, and with these V_i's and n_i's (B) $f_* Eu_W=\sum_i n_i Eu_{V_i}$. ((B) follows step by step in parallel with the proof of (A).) Then the "Chern-Mather" transformation $C_*: \mathcal{F} \to H_*(\ ;Z)$ defined by (for any $X \in \mathrm{Obj}(\mathcal{V})$) $C_*(\sum_W n_W Eu_W):=\sum_W n_W \hat{C}(W)$, where $\hat{C}(W)$ is the Chern-Mather class of W, is in fact nothing but DGM-theory C_* described above. Let us call the property (A) "*total* (*individual*, resp.) pushforward stability" of Chern-Mather class, i.e., the pushforward of the *total* (*individual*, resp.) Chern-Mather class is expressed as a linear combination of *total* (*individual*, resp.) Chern-Mather classes again. Obviously, the "Mather-type" homology class of any linear combination $\sum_{i\geq 0} P_i c_i$ of individual Chern cohomology classes c_i's satisfies "individual pushforward stability". Let us call this linear combination $\sum_{i\geq 0} P_i c_i$ a "*Chern-type*" *characteristic class.*

"*Pushforward Stability Conjecture*": Let ch be a classical characteristic class of vector bundles. The "ch-Mather" homology class $\widehat{Ch}$ (which is defined via the Nash blow-up in the same way as that of the Chern-Mather class) satisfies the "individual pushforward stability" if and only if ch is a "Chern-type" characteristic class.

Remark 1.2. Modulo the above conjecture, a classical characteristic class ch whose "Mather-type" homology class $\widehat{Ch}$ satisfies "total pushforward stability" is a certain "Chern-type" characteristic class $\sum_{i\geq 0} P_i c_i$ with some restrictions on coefficients P_i's.

Proposition 1.3. *Let ch be a "Chern-type" characteristic class and suppose that the "ch-Mather" homology class $\widehat{Ch}$ satisfies "total pushforward stability." Then, $P_0=0$ implies that all the other coefficients P_i's must be equal to zero.*

§ 2. DGM-type theory of characteristic classes.

Let Λ be a commutative integral domain with unit, and let $\mathcal{F}_\Lambda: \mathcal{V} \to \mathcal{A}b$ be the correspondence such that $\mathcal{F}_\Lambda(X)=\mathcal{F}(X)\otimes_Z \Lambda$, where $\mathcal{F}(X)$ is as above. $\mathcal{F}_\Lambda$ clearly becomes a covariant functor as a linear extension of DGM's covariant functor $\mathcal{F}$ with respect to Λ. Using a theorem due to R. Thom ("linear independence of Chern numbers") [6], we can show the following

Theorem 2.1. *Let cl: $Vect \to H^*(\ ;Z)\otimes_Z \Lambda$ be a classical characteristic class of vector bundles with coefficients in Λ. Then, there exists a unique natural transformation $\tau: \mathcal{F}_\Lambda \to H_*(\ ;Z)\otimes_Z \Lambda$ satisfying "smooth condition" that $\tau(X)(1_X)=cl(TX)\cap[X]$ for any smooth variety X if and only if $cl=\lambda c=\lambda+\sum_{i\geq 1}\lambda c_i$, the multiple of the total Chern class c by some element λ of Λ, and $\tau=\lambda C_*$, the multiple of DGM-theory C_* by λ.*

This theorem is a stronger version of Theorem 1.1 (DGM-theory C_*). Thus, we cannot get a unique natural transformation $\tau: \mathcal{F}_{Z[t]} \to H_*(\ ;Z)\otimes_Z Z[t]$ satisfying "smooth condition" that $\tau(X)(1_X)=c_t(TX)\cap[X]$ for any smooth variety X, where c_t is the Chern polynomial. Thus we need to give the correspondence $\mathcal{F}_{Z[t]}$ a different "covariance" structure, i.e., a different pushforward. In [8] we extend DGM-theory C_* to the case of Chern polynomial $c_t:=\sum_{i\geq 0} t^i c_i: K \to H^*(\ ;Z)[t]:=H^*(\ ;Z)\otimes_Z Z[t]$ as follows:

Let $\mathcal{F}^t: \mathcal{CV} \to \mathcal{Ab}$ be the correspondence such that for $X \in \mathrm{Obj}(\mathcal{CV})$ $\mathcal{F}^t(X):=\mathcal{F}(X)\otimes_Z Z[t]$. Then we define our pushforward $f^t_*:=\mathcal{F}^t(f)$ for $f \in \mathrm{Mor}(\mathcal{CV})$ by:

$$f^t_*(Eu_W):=\sum_S n_S t^{\dim W-\dim S} Eu_S,$$

provided that under DGM's topological pushforward f_*

$$f_*(Eu_W)=\sum_S n_S Eu_S.$$

Theorem 2.2. (*"DGM-type" theory C_{t*} of Chern polynomial,* [8]):

(1) *The above correspondence $\mathcal{F}^t$ becomes a covariant functor with the above pushforward,*

(2) *there exists a unique natural transformation $C_{t*}: \mathcal{F}^t \to H_*(\ ;Z)[t]:=H_*(\ ;Z)\otimes_Z Z[t]$ satisfying "smooth condition" that $C_{t*}(1_X)=c_t(TX)\cap[X]$ for any smooth variety X, where 1_X is the characteristic function on X and $c_t(TX)$ is the usual cohomology Chern polynomial of the tangent bundle TX, and*

(3) *(C_{1*}=DGM-theory C_*) "evaluating" C_{t*} at $t=1$ gives DGM-theory C_*.*

Motivated by the formulation of our "DGM-type" theory C_{t*} of Chern polynomial, we define the following:

Definition 2.3. Let $ch: \mathcal{V}ect \to H^*(\ ;Z)\otimes_Z \Lambda$ be a characteristic class of vector bundles. Let $\mathcal{F}^{ch}: \mathcal{CV} \to \mathcal{Ab}$ be the correspondence such that for any $X \in \mathrm{Obj}(\mathcal{CV})$ $\mathcal{F}^{ch}(X):=\mathcal{F}(X)\otimes_Z \Lambda$. Let $Ch_*: \mathcal{F}^{ch} \to H_*(\ ;Z)\otimes_Z \Lambda$ be the "*ch*-Mather" transformation such that for any $X \in \mathrm{Obj}(\mathcal{CV})$ $Ch_*: \mathcal{F}^{ch}(X) \to H_*(X;Z)\otimes_Z \Lambda$ is defined by $Ch_*(\sum_W n_W Eu_W):=\sum_W n_W \widehat{Ch}(W)$, where $\widehat{Ch}(W)$ is the "*ch*-Mather" class of W and $n_W \in \Lambda$. And with this set-up, if we can *make the correspondence $\mathcal{F}^{ch}$ a covariant functor such that Ch_* is a unique natural transformation satisfying "smooth condition" that $Ch_*(1_X)=ch(TX)\cap[X]$ for any smooth variety X,* then we say that the natural transformation $Ch_*: \mathcal{F}^{ch} \to H_*(\ ;Z)\otimes_Z \Lambda$ *is a "DGM-type" theory of the characteristic class ch.*

Remark 2.4. To get such a "DGM-type" theory Ch_* of a characteristic class ch, it is definitely required that the "ch-Mather" homology class $\widehat{Ch}$ should satisfy "total pushforward stability." Hence, by Remark 1.2, we have only to consider a "Chern-type" characteristic class $\sum_{i\geq 0} P_i c_i$.

§3. A characterization of DGM-type theories of characteristic classes. In order to make $\mathcal{F}^{ch}$ a covariant functor we need to define a reasonable pushforward $f_*^{ch} := \mathcal{F}^{ch}(f)$ for any $f: X \to Y$. Motivated by the pushforward in our "DGM-type" theory C_{t*} of the Chern polynomial, we define the following "(in a sense) topologically defined" pushforward:

Definition 3.1. Let ch be a "Chern-type" characteristic class $\sum_{i\geq 0} P_i c_i$ and $f: X \to Y$ and W a subvariety of X. Then

$$f_*^{ch} Eu_W := \sum_S \alpha_S Eu_S, \qquad \text{where } \alpha_S \in \Lambda$$

provided that under DGM's topologically defined pushforward $f_* = \mathcal{F}(f)$,

$$f_* Eu_W = \sum_S n_S Eu_S.$$

Here the "modifier" α_S depends on ch and we need to define α_S so that $\mathcal{F}^{ch}$ becomes a covariant functor with this pushforward.

Remark 3.2. As seen in Proposition 1.3, $P_0 = 0$ implies $ch = 0$. So we can assume that $P_0 \neq 0$. (If $ch = 0$, then no matter how we define the pushforward, we get a stupid DGM-type theory, i.e., $Ch_* = 0$.)

It turns out that without giving a precise definition of the "modifier" α_S, we can show the following "characterization" theorem:

Theorem 3.3. *With Definition* 3.1., *ch has a "DGM-type" theory Ch_* of characteristic class if and only if $ch = h(1 + \sum_{i\geq 1} g^i c_i)$ for some h ($\neq 0$) and $g \in \Lambda$.*

Then, "modulo Ch_*" we can say a little bit more about the "modifier" α_S:

Corollary 3.4. *If Ch_* is a DGM-type theory and each $P_i \neq 0$, then in Definition* 3.1. *the "modifier" α_S is "equal" to $(P_{\dim W}/P_{\dim S}) n_S$ "modulo Ch_*" in the sense that*

$$Ch_*(\sum_S \alpha_S Eu_S) = Ch_*(\sum_S (P_{\dim W}/P_{\dim S}) n_S Eu_S),$$

i.e.,

$$\sum_S \alpha_S \widehat{Ch}(S) = \sum_S (P_{\dim W}/P_{\dim S}) n_S \widehat{Ch}(S).$$

Details of the proofs, related results and topics will appear elsewhere.

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