38. Solution of a Problem of Yokoi

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In [12]–[16] Yokoi studied what he called p-invariants for a real quadratic field $Q(\sqrt{p})$ where $p\equiv 1 \pmod 4$ is prime. In [9] we generalized this concept to an arbitrary real quadratic field $Q(\sqrt{d})$ where d is positive and square-free. We provided numerous applications including bounds for fundamental units and an investigation of the class number one problem related to non-zero n_d, (defined below). It is the purpose of this paper to give a complete list and a proof that the list is valid (with one possible value remaining) of all $Q(\sqrt{d})$ having class number $h(d)=1$ when $n_d\neq 0$. Moreover we show that if the exceptional value of d exists then it is a counterexample to the Generalized Riemann Hypothesis. This completes the task of Yokoi begun in [15]–[16].

In what follows the fundamental unit $\varepsilon_d(>1)$ of $Q(\sqrt{d})$ is denoted $(t_d+u_d\sqrt{d})/\sigma$ where $\sigma=\begin{cases}2 \text{ if } d\equiv 1 \pmod 4\\ 1 \text{ if } d\equiv 2, 3 \pmod 4\end{cases}$. Now set:

$$B=((2t_d)/\sigma-N(\varepsilon_d)-1)u_d^2$$

where N is norm from $Q(\sqrt{d})$. This boundary B was studied in [4], [5] and [14].

The following generalizes Yokoi's notion of a p-invariant n_p where $p\equiv 1 \pmod 4$ is prime (see [12]–[16]).

Let n_d be the *nearest integer* to B; i.e.,

$$n_d=\begin{cases}\lfloor B\rfloor & \text{if } B-\lfloor B\rfloor<\frac{1}{2}\\ \lfloor B\rfloor+1 & \text{if } B-\lfloor B\rfloor>\frac{1}{2}\end{cases}$$

(where $\lfloor x\rfloor$ is the greatest integer less than or equal to x).

In [9] we proved the following:

Theorem 1. *Let $d>0$ be square-free and let $u_d>2$. Then the following are equivalent*:

(1) $n_d=0$

(2) $t_d>4d/\sigma$

(3) $u_d^2>16d/\sigma^2$.

The above generalizes the main result of Yokoi in [12].

We also proved in [9] the following consequences of Theorem 1.

Corollary 1. *If $n_d\neq 0$ then $\varepsilon_d<8d/\sigma^2$.*

Corollary 2. *If $n_d\neq 0$ then there are only finitely many d with $h(d)=1$.*

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Corollary 3. *Let d_0 be a fixed positive square-free integer. Then there are only finitely many d with $u_d = u_{d_0}$ and $h(d)=1$.*

The above generalize results of Yokoi in [13]–[16]. Moreover this has consequences for the Gauss conjecture as follows.

Let:

(G_1): There exist infinitely many real quadratic fields $K=Q(\sqrt{d})$ with $h(d)=1$; (Gauss's conjecture).

(G_2): There exist infinitely many d with $n_d=0$ and $h(d)=1$.

(G_3): For a given natural number n_0 there exists at least one real quadratic field with $h(d)=1$ and $u_d \geq n_0$.

In fact it is easily seen that:

Theorem 2. $(G_1) \leftrightarrow (G_2) \leftrightarrow (G_3)$.

Moreover there are applications for the Artin-Ankeny-Chowla conjecture; that $u_p \not\equiv 0 \pmod p$ if $p \equiv 1 \pmod 4$ is prime; as well as the Mollin-Walsh conjecture [6], that if $d \equiv 7 \pmod 8$ is positive square-free then $u_d \not\equiv 0 \pmod d$. In fact we proved the following in [9].

Theorem 3. *If $d>0$ is square-free and $n_d \neq 0$ then* $u_d \not\equiv 0 \pmod d$.

Thus the aforementioned two conjectures hold when $n_d \neq 0$.

Now we turn to the main function of this paper which is to use the above results to actually determine all d with $h(d)=1$ and $n_d \neq 0$.

First we provide a table of such values, and then prove that we have all of them, (except possibly one which we show would be a counter-example to the Generalized Riemann Hypothesis).

Theorem 4. *If $h(d)=1$ and $n_d \neq 0$ then (with possibly one more value remaining) d is an entry in the following Table.*

Table

d	$\log(\varepsilon_d)$	d	$\log(\varepsilon_d)$	d	$\log(\varepsilon_d)$	d	$\log(\varepsilon_d)$
2	.881373587	53	1.9657204716	237	4.3436367167	917	7.0741160992
3	1.866264041	61	3.6642184609	269	5.0999036060	941	7.0343887062
5	0.4812118251	62	4.8362189128	293	2.8366557290	1013	6.8276304083
6	2.2924316696	69	3.2172719712	317	4.4887625925	1077	5.8888702849
7	2.7686593833	77	2.1846437916	341	5.6240044731	1133	4.6150224728
11	2.9932228461	83	5.0998292455	398	6.6821070271	1253	5.1761178117
13	1.1947632173	93	3.3661046429	413	4.1106050108	1293	7.4535615360
14	3.4000844141	101	2.9982229503	437	3.0422471121	1493	7.7651450829
17	2.0947125473	133	5.1532581804	453	5.0039012599	1613	7.9969905191
21	1.5667992370	141	5.2469963702	461	5.8999048596	1757	6.9137363626
23	3.8707667003	149	4.1111425009	509	6.8297949062	1877	7.3796325418
29	1.6472311464	157	5.3613142065	557	5.4638497592	2453	8.1791997198
33	3.8281684713	167	5.8171023021	573	6.6411804655	2477	6.4723486834
37	2.4917798526	173	2.5708146781	677	3.9516133361	2693	8.3918567515
38	4.3038824281	197	3.3334775869	717	5.4847797157	3053	8.1550748053
41	4.1591271346	213	4.2902717358	773	4.9345256863	3317	8.5642675624
47	4.5642396669	227	6.1136772851	797	5.9053692725	3533	7.7985232220

Proof. By Corollary 1 we have that $\varepsilon_d < 8d/\sigma^2$. Thus our task is to find all positive square-free d such that $h(d)=1$ and $1 < \varepsilon_d < 8d/\sigma^2$. Let $\Delta = 4d/\sigma^2$. A classical class number formula is:

$$2h(d)\log(\varepsilon_d) = \sqrt{\Delta}\, L(1, \chi).$$

Moreover a result of Tatuzawa [11] says:

If $\frac{1}{2} > \alpha > 0$ and $\Delta \geq \max(e^{1/\alpha}, e^{11.2})$ then with one possible exception $L(1, \chi) > 0.655\alpha/\Delta^{\alpha}$ where χ is a real, non-principal, primitive character modulo Δ.

We now use the above to complete our task.

Choose $\alpha = .0885$ and $\Delta > 80{,}775.9$. Then, since $\log \varepsilon_d < \log 2\Delta$ we have; (with one possible exception):

$$h(d) > (\sqrt{\Delta})(.0885)(.655)/(2\log 2\Delta)(\Delta^{.0885}).$$

Hence $h(d) > 1$ if $\Delta > 5 \times 10^6$; (in fact $h(d) > 1.026755418$).

Now we proceed to show that below this bound the only $h(d)=1$ with $\varepsilon_d < 8d/\sigma^2$ are those in the Table. First we need some notation and facts from the theory of continued fractions.

Let $w_d = (\sigma - 1 + \sqrt{d})/\sigma$ and denote the continued fraction of w_d by $w_d = \langle a, \overline{a_1, a_2, \cdots, a_k} \rangle$; whence having period k; and $a_0 = 1 = \lfloor w_d \rfloor$ while:

$$a_i = \lfloor (P_i + \sqrt{d})/Q_i \rfloor \text{ for } i \geq 1,$$

where:

$$(P_0, Q_0) = (\sigma - 1, \sigma);\ P_{i+1} = a_i Q_i - P_i \quad \text{for } i \geq 0 \text{ and}$$
$$Q_{i+1} Q_i = d - P_{i+1}^2 \quad \text{for } i \geq 0.$$

Now we return to our task.

Case 1. $d \equiv 2, 3 \pmod 4$; whence $\Delta = 4d$.

Since Δ is even then 2 ramifies. Thus by [3, Theorem 2.1], $Q_{k/2} = 2$, with k even whenever $h(d)=1$, provided $\Delta > 20$. (If $\Delta \leq 20$ then we get our values $d = 2, 3$ of the Table).

From [7] we also have:

$$\varepsilon_d = \prod_{i=1}^{k} (P_i + \sqrt{d})/Q_{i-1} \quad (P_i \geq 1).$$

Thus:

$$\varepsilon_d > (\sqrt{d})(\sqrt{d}/2) \prod (P_i + \sqrt{d})/Q_{i-1}$$

where the product runs from $i=2$ to $i=k$, *excluding* $i = k/2 + 1$.

Now;

$$((P_i + \sqrt{d})/Q_{i-1})((P_{i+1} + \sqrt{d})/Q_i)$$
$$= (P_{i+1} + \sqrt{d})/(\sqrt{d} - P_i) = a_i Q_i/(\sqrt{d} - p_i)) + 1 > 2.$$

If $k \geq 10$ then $\varepsilon_d > (\sqrt{d})(\sqrt{d}/2) 2^{(k/2)-1} \geq 8d$, a contradiction. Since $k \leq 10$ then by computation we arrive at $d \leq 7653$. Our computation shows that of those values only the following satisfy our criteria and appear in the Table:

$$d \in \{2, 3, 6, 7, 11, 14, 23, 38, 47, 62, 83, 167, 227, 398\}.$$

Case 2. $d = \Delta \equiv 1 \pmod 8$.

Thus 2 splits and so since $h(d)=1$ we get $Q_i/2 = 2 = Q_{(k-j)}/2$ for some $j \neq 0$ (provided $d > 20$). (If $d \leq 20$ then we get only the value $d = 17$ which is on our Table).

Therefore:

$$\varepsilon_d > (\sqrt{d}/2)(\sqrt{d}/4)^2 \prod (P_i+\sqrt{d})/Q_{i-1} \geq d\sqrt{d}/32 > 2d$$

when $\sqrt{d}>64$, a contradiction. (Here the product runs from $i=2$ to $i=k$ *excluding* $i=j+1$ *and* $i=k-j+1$.)

Hence $\sqrt{d}\leq 64$; i.e., $d<4096$. In this range our computation gives us only the following values satisfying our criteria: $d\in\{17, 33, 41\}$.

Case 3. $d=\Delta\equiv 5 \pmod 8$.

By [2], since $\Delta<5\times 10^6$ there exists a prime $p\leq 67$ such that $(\Delta/p)=1$, where $(/)$ is the Kronecker symbol. Suppose $\sqrt{\Delta}/2>67$. Then p splits in $Q(\sqrt{\Delta})$ and so $Q_j=Q_{k-j}=2p$ for some $j\neq 0$ (provided $\Delta>20$. If $\Delta\leq 20$ then we get only $d=5, 13$).

Now let $\tau=(1+\sqrt{5})/2$ and $\psi_i=(P_i+\sqrt{d})/Q_{i-1}$. By [10, Corollary 1, p. 873] $\prod_{i=a}^{b}\psi_i>\tau^{b-a}$ for $b\geq a$. Thus:

$$\prod \psi_i=\prod_{i=2}^{j}\psi_i \prod_{i=j+2}^{k-j}\psi_i \prod_{i=k-j+2}^{k}\psi_i>\tau^{j-2+k-j-(j+2)+k-(k-j+2)}=\tau^{k-6}$$

(where the initial product ranges over $i=2$ to $i=k$ *excluding* $i=j+1$ *and* $i=k-j+1$).

Hence $\varepsilon_d>(\sqrt{d}/2)(\sqrt{d}/2p)^2\prod\psi_i>2d(\sqrt{d}\,\tau^{k-6}/16p^2)$ where the product ranges as in the previous one. Since $p\leq 67$ we get that if $\tau^{k-6}>536$ then: $\sqrt{d}\,\tau^{k-6}>71824>16p^2$. But $\tau^{k-6}>536$ implies $k-6>(\log 536)/\log\tau\approx 13.06$ so $k>19.06$. Thus: If $d>17956$ and $k\geq 20$ then $\varepsilon_d>2d$, a contradiction. If $d>17956$ and $k<20$ then $h(d)=1$ implies by computation that $d\leq 30917$. In this case there exists a prime $p\leq 29$ such that $(d/p)=1$. Hence if $\sqrt{d}>2\cdot 29=58$ we get $Q_j=Q_{k-j}=2p$ for some $p\leq 29$. Thus $\varepsilon_d>2d(\sqrt{d}\,\tau^{k-6}/16\cdot 29^2)$ as above. Hence, if $d>13456$ and $k\geq 16$ then $\varepsilon_d>2d$, a contradiction. If $d>13456$ and $k\leq 15$ then $d<23117$. Our computation on this bound now yields the remaining values in the Table.

Remark 1. In [15] Yokoi found the 30 primes $p\equiv 1 \pmod 4$ with $h(p)=1$ and $n_p\neq 0$ (with one possible exception). We have completed the task by adding another 38 values to the list for a total of 68. As seen by the above proof there are 14 values of $d\equiv 2, 3 \pmod 4$ of which 9 are primes. For $d\equiv 1 \pmod 8$ we got only 17, 33 and 41. The remainder are $d\equiv 5 \pmod 8$. Of these 51 remaining values 28 are primes, those found by Yokoi along with 17 and 41. The composite values which we added are the 23 values:

{21, 69, 77, 93, 133, 141, 213, 237, 341, 413, 437, 453, 573, 717,
917, 1077, 1133, 1253, 1293, 1757, 2453, 3053, 3317}.

We also have a list, too long to include here, of *all* values of squarefree d with $n_d\neq 0$, up to 39,999 with their class numbers and regulators.

Remark 2. Kim [1] has shown that if the Generalized Riemann Hypothesis (GRH) holds then Tatuzawa's theorem is true without exception. Hence if the exceptional value exists then it is a counterexample to the GRH.

Remark 3. Observe that the Table contains all the ERD-types with $h(d)=1$ (i.e., all types $h(d)=1$ where $d=l^2+r$ with $4l\equiv 0 \pmod r$). These were found by the authors in [8]. Thus there are 25 *non*-ERD type and they are

{41, 61, 133, 149, 157, 269, 317, 341, 461, 509, 557, 773, 797, 917, 941, 1013, 1493, 1613, 1877, 2453, 2477, 2693, 3053, 3317, 3533}.

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