32. *Regular Duo Near-rings*

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1. Introduction. In ring theory, it is well known that regular duo rings are characterized in terms of quasi-ideals (see [1, 3, 4]). The purpose of this note is to extend the above result to a class of regular duo near-rings. As to terminology and notation, we follow the usage in [2].

2. Preliminaries. Let N be a near-ring, which always means right one throughout this note.

If A, B and C are three non-empty subsets of N, then AB (ABC) denotes the set of all finite sums of the form $\sum a_k b_k$ with $a_k \in A$, $b_k \in B$ ($\sum a_k b_k c_k$ with $a_k \in A$, $b_k \in B$, $c_k \in C$).

A *right N-subgroup* (*left N-subgroup*) of N is a subgroup H of $(N, +)$ such that $HN \subseteq H$ ($NH \subseteq H$). A *quasi-ideal* of a zero-symmetric near-ring N is a subgroup Q of $(N, +)$ such that $QN \cap NQ \subseteq Q$. Right N-subgroups and left N-subgroups are quasi-ideals. The intersection of a family of quasi-ideals is again a quasi-ideal.

An element n of N is said to be *regular* if $n = nxn$ for some $x \in N$, and N is called *regular* if every element of N is regular.

Lemma 1. *Let N be a regular zero-symmetric near-ring. Then the following assertions hold:*

(i) *For every quasi-ideal Q of N, $Q = QNQ = QN \cap NQ$.*

(ii) *For every right N-subgroup R and left N-subgroup L of N, $RL = R \cap L$.*

Proof. (i) Let Q be a quasi-ideal of N, that is, $QN \cap NQ \subseteq Q$. By the regularity of N, $Q \subseteq QNQ$. Moreover we have $QNQ \subseteq QN$ and $QNQ \subseteq NQ$. Hence it follows that $Q \subseteq QNQ \subseteq QN \cap NQ \subseteq Q$. Thus $Q = QNQ = QN \cap NQ$.

(ii) Let R and L be right and left N-subgroups of N, respectively. Then $RL \subseteq R \cap L$ always holds. So we have to show only that an arbitrary element n of the intersection $R \cap L$ lies in RL. By the regularity of the element n, there exists an x in N such that $n = nxn$. Since $n \in R$ and $xn \in L$, we have $n = nxn \in RL$.

For an element n of a near-ring N, $(n)_r$ ($(n)_l$) denotes the principal right (left) N-subgroup of N generated by n, and $[n]$ denotes the subgroup of $(N, +)$ generated by n.

Lemma 2. *Let N be a near-ring with identity and n an element of N. Then $(n)_r = [n]N$ and $(n)_l = Nn$.*

Proof. Since N has the identity, $n \in [n]N$. Moreover $[n]N$ is a right N-subgroup of N. So $(n)_r \subseteq [n]N$. On the other hand, the inclusion $[n]N \subseteq (n)_r$ always holds. Hence we have $(n)_r = [n]N$. Similarly $(n)_l = Nn$.

A near-ring N is said to be an *S-near-ring*, if $n \in Nn$ for every element n of N.

Lemma 3 ([5, Theorem]). *The following conditions on a zero-symmetric near-ring N are equivalent*:

(a) *N is regular and has no non-zero nilpotent elements.*

(b) *N is an S-near-ring, and for any two left N-subgroups L_1, L_2 of N, $L_1 \cap L_2 = L_1 L_2$.*

3. Duo near-rings. A near-ring N is called a *duo near-ring* if every one-sided (right or left) N-subgroup of N is a two-sided N-subgroup of N.

Proposition 1. *Every duo near-ring is zero-symmetric.*

Proof. Let N be a duo near-ring. Since $\{0\}$ is a right N-subgroup of N, the assumption implies that $\{0\}$ is also a left N-subgroup of N. So $N\{0\} \subseteq \{0\}$. Thus N is zero-symmetric.

An element n of a near-ring N is said to be a *duo element* of N, if the principal right N-subgroup $(n)_r$ and the principal left N-subgroup $(n)_l$ of N generated by n are equal: $(n)_r = (n)_l$.

Proposition 2. *A near-ring N is a duo near-ring if and only if every element of N is a duo element.*

Proof. If N is a duo near-ring, then evidently every element of N is duo.

Conversely, assume that every element of N is duo. Let R be an arbitrary right N-subgroup of N. Consider an element n of N and an element x of R. Evidently, the principal right N-subgroup $(x)_r$ of N generated by x is contained in R. As x is a duo element, we get

$$nx \in N(x)_r = N(x)_l \subseteq (x)_l = (x)_r \subseteq R,$$

that is, R is a left N-subgroup of N. Similarly one can show that any left N-subgroup of N is also a right N-subgroup of N.

4. Regular duo near-rings. Now we state the main results of this note.

Theorem 1. *The following conditions on a near-ring N are equivalent*:

(1) *N is a regular duo near-ring.*

(2) *N is an S-near-ring, and for any two quasi-ideals Q_1, Q_2 of N, $Q_1 Q_2 = Q_1 \cap Q_2$.*

(3) *N is an S-near-ring, and for any two left N-subgroups, L_1, L_2 and right N-subgroups R_1, R_2 of N, $L_1 L_2 = L_1 \cap L_2$ and $R_1 R_2 = R_1 \cap R_2$.*

(4) *N is an S-near-ring, and for every left N-subgroup L and right N-subgroup R of N, $L \cap R = LR$.*

Proof. (1)$\Rightarrow$(2): Clearly every regular near-ring is an S-near-ring. Suppose that N is a regular duo near-ring. Then, by Proposition 1, N is zero-symmetric.

We first show that every quasi-ideal of N is a two-sided N-subgroup of N. In fact, let Q be any quasi-ideal of N. By Lemma 1-(i), we have $Q=QNQ$. Moreover, since QN is a right N-subgroup of N, QN is also a left N-subgroup of N. So we have $NQ=N(QNQ)\subseteq N(QN)\subseteq QN$. This and Lemma 1-(i) imply $Q=QN\cap NQ=NQ$, whence Q is a left N-subgroup of N. Hence Q is a two-sided N-subgroup of N.

Now, let Q_1, Q_2 be any two quasi-ideals of N. Then, by the above result, Q_1 is a right N-subgroup of N and Q_2 is a left N-subgroup of N. Hence, Lemma 1-(ii) implies $Q_1Q_2=Q_1\cap Q_2$.

The implications (2)$\Rightarrow$(3) and (2)$\Rightarrow$(4) are evident.

(3)$\Rightarrow$(1): Let L_1 be any left N-subgroup of N, and let $L_2=N$. Then the assumption (3) implies $L_1N=L_1\cap N=L_1$, that is, L_1 is a right N-subgroup of N. Similarly one can show that any right N-subgroup is also a left N-subgroup of N. This means that N is a duo near-ring.

On the other hand, since a duo near-ring is zero-symmetric by Proposition 1, the assumption (3) and Lemma 3 imply that N is regular. Thus N is a regular duo near-ring.

The implication (4)$\Rightarrow$(1) can be proved similarly.

Theorem 2. *Let N be a zero-symmetric near-ring with identity. Then the following conditions are equivalent:*

(I) *N is a regular near-ring without non-zero nilpotent elements.*

(II) *N is a regular duo near-ring.*

(III) *For any two quasi-ideals Q_1, Q_2 of N, $Q_1\cap Q_2=Q_1Q_2$.*

Proof. (I)$\Rightarrow$(II): In view of Proposition 2, it is enough to prove that every element n of N is duo, that is, $(n)_r=(n)_l$.

By the condition (I), any element n of N is regular, that is, $n=nxn$ for some x in N. This implies

$$nx=e=e^2,\quad xn=f=f^2,\quad n=en=nf,$$

whence $(n)_r=(e)_r$ and $(n)_l=(f)_l$.

On the other hand, by [2, 9.43 Proposition], the idempotent elements e and f are central. So, by Lemma 2, we get

$$(e)_l=Ne\subseteq eN\subseteq(e)_r,\quad (f)_r=[f]N\subseteq Nf=(f)_l.$$

Since $n=en=ne\in Ne=(e)_l$ and $n=nf=fn\in[f]N=(f)_r$, we get

$$(n)_l\subseteq(e)_l\quad\text{and}\quad(n)_r\subseteq(f)_r.$$

Hence it follows that

$$(n)_l\subseteq(e)_l\subseteq(e)_r=(n)_r\quad\text{and}\quad(n)_r\subseteq(f)_r\subseteq(f)_l=(n)_l,$$

whence $(n)_r=(n)_l$.

The implications (II)$\Rightarrow$(III) and (III)$\Rightarrow$(I) follow from Theorem 1 and Lemma 3, respectively.

5. **Remark.** In Theorem 2, the implication (II)$\Rightarrow$(I) is true without the assumption that N has the identity, because by the implication (1) $\Rightarrow$ (3) in Theorem 1 the condition (b) of Lemma 3 always holds for a regular duo near-ring. However, the following example shows that the converse does

not hold in general.

Let $N=\{0,1,2,3\}$ be the near-ring due to [2, Near-rings of low order (E-1)] defined by the tables

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

·	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	2	2	2
3	0	3	3	3

Then N is a regular zero-symmetric near-ring without non-zero nilpotent elements. But N is not duo. In fact, a right N-subgroup $\{0,1\}$ of N is not a left N-subgroup of N, since $N\{0,1\}=N$.

References

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