1. *Functorial Properties of Second Analytic Wave Front Sets and Equivalence of Two Notions of Second Microlocal Singularities*

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1. Introduction. This paper aims at studying several functorial properties of second analytic wave front sets of hyperfunctions or microfunctions. As an application we show that two notions of second microlocal singularities, second singular spectrum introduced by M. Kashiwara and second analytic wave front sets due to J. Sjöstrand, are equivalent. To this aim we utilize the partial Radon transformation. We follow the notation prepared in Okada-Tose [11].

2. Second singular spectrum. Let M be, as in §2 of [11], an open subset in $\boldsymbol{R}^n_x$, and X be a complex neighborhood of M in $\boldsymbol{C}^n_z$. We take coordinates of $T^*_MX(\simeq\sqrt{-1}\,T^*M)$ [resp. T^*M] as $(x;\sqrt{-1}\xi\cdot dx)$ [resp. $(x;\xi\cdot dx)$] with $\xi=(\xi_1,\cdots,\xi_n)$. We identify $\sqrt{-1}\,T^*M$ with T^*M by the correspondence

(2.1) $$(x;\sqrt{-1}\xi\cdot dx)\longleftrightarrow(x;\xi\cdot dx).$$

T^*_MX is endowed with the sheaf $\mathcal{C}_M$ of microfunctions, which enjoys an exact sequence

$$0\longrightarrow\mathcal{A}_M\longrightarrow\mathcal{B}_M\longrightarrow\dot{\pi}_{M*}(\mathcal{C}_M|_{T^*_MX\setminus M})\longrightarrow 0.$$

Here $\mathcal{A}_M$ denotes the sheaf of real analytic functions on M, $\mathcal{B}_M$ that of hyperfunctions, and $\dot{\pi}_M$ the restriction to $\dot{T}^*_MX$ $(\simeq T^*_MX\setminus M)$ of the natural projection $\pi_M: T^*_MX\to M$. Moreover there exists a canonical sheaf morphism

$$Sp_M: \pi_M^{-1}\mathcal{B}_M\longrightarrow\mathcal{C}_M\qquad(\pi_M: T^*_MX\longrightarrow M),$$

by which we set for $u\in\mathcal{B}_M$

$$\mathrm{SS}\,(u):=\mathrm{supp}\,(Sp_M(u)).$$

Then $\mathrm{SS}\,(u)$ is called the singular spectrum of u (refer to [13]). We remark that for $u\in\mathcal{B}_M$, we have

$$\mathrm{SS}\,(u)=\mathrm{WF}_a\,(u)$$

through the correspondence (2.1). This is a classical fact dating back to J.M. Bony [2], K. Kataoka [7]. Refer also to J. Sjöstrand [12].

Now let V denote an involutive submanifold in T^*_MX:

$$V=\{(x;\sqrt{-1}\xi\cdot dx);\ \xi_1=\cdots=\xi_d=0\}.$$

We set

$$N=\{z\in X;\ \mathrm{Im}\,z''=0\},\qquad \tilde{V}=T^*_NX\setminus N.$$

We take coordinates of $\tilde{V}$ as $(z',x'';\sqrt{-1}\xi''\cdot dx'')$, and we have in T^*X the injection

$$V=\dot{T}^*_MX\cap\dot{T}^*_NX\hookrightarrow\tilde{V}.$$

$\tilde{V}$ is endowed with the sheaf $\mathcal{C}_{\tilde{V}}$ of microfunctions with holomorphic parameters z' (cf. [13; Chap. 3]). M. Kashiwara [4] (cf. J.M. Bony [1]) constructed the sheaf $\mathcal{C}_V^2$ of 2-microfunctions on $T_V^*\tilde{V}$ from the sheaf $\mathcal{C}_{\tilde{V}}$ by the same procedure as M. Sato *et al.* [13] constructed the sheaf $\mathcal{C}_M$ from $\mathcal{O}_X$. Explicitly, the sheaf $\mathcal{C}_V^2$ is given by

(2.2) $$\mathcal{C}_V^2=\mu_V(\mathcal{C}_{\tilde{V}})[d]$$

where μ_V denotes the functor of Sato's microlocalization along V (refer to [6]). We also set

(2.3) $$\mathcal{B}_V^2=\mathcal{C}_V^2|_V=\boldsymbol{R}\Gamma_V(\mathcal{C}_{\tilde{V}})[d].$$

Remark that the complexes in (2.2) and (2.3) are concentrated in degree 0. Moreover there exist the exact sequences

(2.4) $$0\longrightarrow\mathcal{A}_V^2\longrightarrow\mathcal{B}_V^2\longrightarrow\dot{\pi}_{V*}(\mathcal{C}_V^2|_{\dot{T}_V^*\tilde{V}})\longrightarrow 0,$$

(2.5) $$0\longrightarrow\mathcal{C}_M|_V\longrightarrow\mathcal{B}_V^2.$$

Here we set $\mathcal{A}_V^2=\mathcal{C}_{\tilde{V}}|_V$, and $\dot{\pi}_V$: $\dot{T}_V^*\tilde{V}\to V$. There also exists a canonical morphism

$$Sp_V^2:\pi_V^{-1}\mathcal{B}_V^2\longrightarrow\mathcal{C}_V^2\qquad(\pi_V:T_V^*\tilde{V}\longrightarrow V),$$

by which we define for $u\in\mathcal{B}_V^2$ (in particular for $u\in\mathcal{C}_M|_V$) its second singular spectrum along V by

$$\mathrm{SS}_V^2(u):=\mathrm{supp}\,(Sp_V^2(u)).$$

We give several remarks: i) We identify T_VT^*M with $T_V^*\tilde{V}$ through the correspondence

(2.6) $$(x;\xi''\cdot dx'';x'^*\cdot\partial/\partial\xi')\longleftrightarrow(x;\sqrt{-1}\xi''\cdot dx'';\sqrt{-1}x'^*\cdot dx').$$

This is intrinsic if we admit (2.1). ii) For $\dot{q}=(\dot{x};\sqrt{-1}\dot{\xi}''\cdot dx'')$ and $u\in\mathcal{C}_M|_{V,\dot{q}}$, $u\in\mathcal{A}_{V,\dot{q}}$ if and only if $\dot{q}\notin$ 2-singsupp$_V(u)$ in the sense of Definition 2.1 of [11]. This is essentially shown in J.M. Bony [3]. iii) Refer to Kashiwara-Laurent [5] for more details about $\mathcal{C}_V^2$.

3. Functorial properties of second analytic wave front sets. To simplify the notation, we set for $u\in\mathcal{B}_M(M)$

$$\widehat{\mathrm{WF}}_{a,V}^2(u)=\mathrm{WF}_{a,V}^2(u)\cup(\mathrm{WF}_a(u)\cap V)$$

in $T_V\dot{T}^*M$, and

$$\widehat{\mathrm{WF}}_a(u)=\mathrm{WF}_a(u)\cup\mathrm{supp}\,(u).$$

By several variants of Theorem 3.4 in Okada-Tose [11], we can develop the study of functorial properties of second analytic wave front sets. We follow the notation prepared in § 2 of [11].

3.1. Tensor product. Let M_1 be an open subset of $\boldsymbol{R}^l$ with coordinates $t=(t_1,\cdots,t_l)$. We take coordinates of T^*M_1 as $(t;\tau\cdot dt)$ with $\tau\in\boldsymbol{R}^l$. Moreover we define an involutive submanifold V_1 in $\dot{T}^*(M\times M_1)$ by

$$V_1=\{(x,t;\xi\cdot dx+\tau\cdot dt);\xi'=0,\tau=0\},$$

and we take also a system of coordinates of $T_{V_1}T^*(M\times M_1)$ as $(x,t;\xi''\cdot dx'';x'^*\cdot\partial/\partial\xi'+t^*\cdot\partial/\partial\tau)$ with $t^*=(t_1^*,\cdots,t_l^*)$. Here we give

Theorem 3.1. *Let $u(x)\in\mathcal{B}_M(M)$ and $v(t)\in\mathcal{B}_{M_1}(M_1)$. Then*

$$\mathrm{WF}_{a,V_1}^2(u(x)\cdot v(t))\subset\{(x,t;\xi'';x'^*,t^*)\in\dot{T}_{V_1}\dot{T}^*(M\times M_1);(x;\xi'';x'^*)\in\widehat{\mathrm{WF}}_{a,V}^2(u(x)),\ (t;t^*\cdot dt)\in\widehat{\mathrm{WF}}_a(v(t))\}.$$

3.2. Restriction. We follow the notation prepared in § 2 of [11]. We set

$$M_2=\{x\in M\,;\,x_1=0\}$$

and take coordinates of M_2 (resp. T^*M_2) as (t, x'') (resp. $(t, x''\,;\,\tau\cdot dt+\xi''\cdot dx'')$) with $t=(x_2, \cdots, x_d)$ (resp. $\tau=(\xi_2, \cdots, \xi_d)$). We define an involutive submanifold V_2 by

$$V_2=\{(t, x''\,;\,\tau, \xi'')\in \dot{T}^*M_2\,;\,\tau=0\}.$$

In this situation, we give

Theorem 3.2. *Let $u(x)$ be a hyperfunction defined in M. We assume*

$$\mathrm{WF}_a(u)\cap\{(\dot{x}\,;\,\pm dx_1)\in T^*M\}=\varnothing, \tag{3.1}$$

$$(0, \dot{t}, \dot{x}''\,;\,\xi_1, 0, \cdots, 0, \dot{\xi}'')\notin \mathrm{WF}_a(u) \tag{3.2}$$

for any $\xi_1\in \boldsymbol{R}\setminus\{0\}$. *Moreover we assume that*

$$(0, \dot{t}, \dot{x}''\,;\,\dot{\xi}''\,;\,\pm\partial/\partial\xi_1)\notin \mathrm{WF}^2_{a,V}(u). \tag{3.3}$$

Then

$$\mathrm{WF}^2_{a,V_2}(u|_{x_1=0})\cap\{(\dot{t}, \dot{x}''\,;\,\dot{\xi}''\,;\,t^*\cdot\partial/\partial\tau)\in \dot{T}_{V_2}\dot{T}^*M_2\}$$
$$\subset\{(\dot{t}, \dot{x}''\,;\,\dot{\xi}''\,;\,t^*\cdot\partial/\partial\tau)\,;\,\exists x_1^* \text{ such that } (0, \dot{t}, \dot{x}''\,;\,\dot{\xi}''\,;\,(x_1^*, t^*))\in \mathrm{WF}^2_{a,V}(u(x))\}.$$

3.3. Integration along fibers. Let M be $\boldsymbol{R}^n_x$, and V be as defined in § 2 of [11]. We set $M_3=\boldsymbol{R}^{n-1}$ endowed with the coordinates (t, x'') $(t=(x_2, \cdots, x_d))$. We define an involutive submanifold V_3 in $\dot{T}^*M_3$ by

$$V_3=\{(t, x''\,;\,\tau\cdot dt+\xi''\cdot dx'')\in \dot{T}^*M_3\,;\,\tau=0\}.$$

Moreover f denotes the projection

$$f: M\longrightarrow M_3 \qquad x\longmapsto(t, x'').$$

Then we give

Theorem 3.3. *Let $u(x)\in \mathcal{C}_M$. We assume that the natural projection*

$$\varpi: M\times_{M_3}T^*M_3\longrightarrow T^*M_3$$

is proper on $\mathrm{supp}(u)$. *Then*

$$\mathrm{WF}^2_{a,V_3}\left(\int u(x)dx_1\right)\subset\{(t, x''\,;\,\xi''\cdot dx''\,;\,t^*\cdot\partial/\partial\tau)$$
$$\in \dot{T}_{V_3}\dot{T}^*M_3\,;\,\exists x_1 \text{ such that } (x_1, t, x''\,;\,\xi''\,;\,(0, t^*))\in \mathrm{WF}^2_{a,V}(u)\}.$$

3.4. Equivalence of two definitions of second microlocal singularities. We follow the notation prepared in § 2. We define a hyperfunction on $\boldsymbol{R}_n\times S^{d-1}$

$$W(x', x'^*)=\frac{(d-1)!}{(-2\pi\sqrt{-1})^d}$$
$$\times\frac{(1-\sqrt{-1}x'\cdot x'^*)^{d-1}-(1-\sqrt{-1}x'\cdot x'^*)^{d-2}(x'^2-(x'\cdot x'^*)^2)}{\{x'\cdot x'^*+\sqrt{-1}(x'^2-(x'\cdot x'^*)^2)+\sqrt{-1}0\}^d}.$$

Now we give

Theorem 3.4. *Let $u(x)$ be a microfunction defined in a neighborhood of $p_0=(\dot{x}\,;\,\sqrt{-1}\dot{\xi}''\cdot dx'')\in V$. Then the following conditions* (i), (ii), (iii) *are equivalent to one another*:

(i) $\dot{q}=(\dot{x}\,;\,\dot{\xi}''\,;\,\dot{x}'^*)\notin \mathrm{WF}^2_{a,V}(u)$,

(ii) *the microfunction*

$$v(x, x'^*)=\int u(\tilde{x}', x'')W(x'-\tilde{x}', x'^*)d\tilde{x}'$$

has holomorphic parameters (x', x'^*) *in a neighborhood of* $p_1 = (\dot{x}, \dot{x}'^*; \sqrt{-1}\dot{\xi}'' \cdot dx'')$,

(iii) $\dot{q} = (\dot{x}; \sqrt{-1}\dot{\xi}''; \sqrt{-1}\dot{x}'^*) \notin SS_V^2(u)$.

The equivalence between (ii) and (iii) is already shown in [9; Proposition 7.4]. We use results in § 3.1–§ 3.3 to show that (i) and (ii) are equivalent. Refer to Okada-Tose [10] for details.

By the above theorem, we have for $u \in \mathcal{C}_M|_V$

$$SS_V^2(u) = WF_{a,V}^2(u)$$

through the correspondence (2.6).

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