26. On Algebroid Solutions of Some Algebraic Differential Equations in the Complex Plane

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1. Introduction. The purpose of this paper is to generalize the results obtained for binomial differential equations ([9]) to general algebraic differential equations. Let a_{jk} $(j=0, 1, \cdots, n;\ k=0, 1, \cdots, q_j)$ be entire functions without common zero for which $a_{0q_0}\cdot a_{nq_n}\neq 0$. Put

$$Q_j(z, w)=\sum_{k=0}^{q_j} a_{jk}w^k,\ \ (q_j=\deg Q_j)$$

and we consider the differential equation (=D.E.)

$$(1)\qquad \sum_{j=1}^{n} Q_j(z, w)(w')^j=Q_0(z, w)$$

under the condition

$$(2)\qquad q_n+n>q_j+j\ \ (j=1, 2, \cdots, n-1).$$

We suppose that the D.E. (1) is irreducible over the field of meromorphic functions in $|z|<\infty$ and that it admits at least one nonconstant ν-valued algebroid solution $w=w(z)$ in the complex plane. We say that the solution w is *admissible* if

$$T(r, f/a_{nq_n})=o(T(r, w))$$

for $r\to\infty$, possibly outside a set of finite linear measure, where $f=a_{jk}$ $(j=0, 1, \cdots, n;\ k=0, 1, \cdots, q_j)$. For example, when all a_{jk} are polynomials, a transcendental algebroid solution of the D.E. (1) is admissible.

In this paper we denote by E a subset of $[0, \infty)$ for which $m(E)<\infty$ and by K a positive constant. E or K does not always mean the same one when they will appear in the following. Further, the term "algebroid" (resp. "meromorphic") will mean algebroid (resp. meromorphic) in the complex plane. We use the standard notation of the Nevanlinna theory of meromorphic ([3]) or algebroid functions ([6], [10], [11]).

2. Lemmas. In this section, we shall give three lemmas for later use.

Lemma 1. *Let v be a transcendental algebroid function such that v and v' have at most a finite number of poles. Then, for some positive constants K_1 and K_2 it holds*

$$M(r, v)\leqq K_1+K_2rM(r, v')\quad (r\notin E),$$

where $M(r, v)=\max_{|z|=r}|v(z)|$ ([5]).

Lemma 2. *Let g be a transcendental entire function. Then,*

$$M(r, g')\leqq 2M(r, g)^2\quad (r\notin E)\qquad ([4]).$$

Lemma 3. *The absolute values of roots of the equation*

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

are bounded by

$$\max\{n|a_1|, (n|a_2|)^{1/2}, \cdots, (n|a_n|)^{1/n}\} \qquad ([7]).$$

3. **Theorems.** Let $w = w(z)$ be a nonconstant v-valued algebroid solution of the D.E. (1) under the condition (2). We put

$$q_n = q \quad \text{and} \quad \max_{0 \leqq j \leqq n-1} \{q_j + j\} = p$$

for simplicity.

Theorem 1. I) *If $q_0 \leqq q + n$, then the poles of w are contained in the set of zeros of a_{nq}.*

II) *If $p < q + n$, then*

$$N(r, w) \leqq KN(r, 1/a_{nq}).$$

We can prove this theorem as in the case of Theorem 1 in [9] by means of the test-power test.

Theorem 2. *Suppose that $p < q + n$ and a_{nq} is a polynomial. Then, $w = w(z)$ satisfies*

$$\min(n, q+n-p)\log^+ M(r, w) \leqq K \sum_{j,k} \log^+ M(r, a_{jk}) + O(\log r) \quad (r \notin E).$$

Proof. If w is algebraic, there is nothing to prove. We then suppose that w is transcendental and $M(r, w) \geqq 1$ $(r \notin E)$. Let S be the set of zeros of a_{nq}. S is then a finite set and the poles of w are contained in it by Theorem 1-I). Since w is a solution of the D.E. (1), it satisfies

$$(3) \qquad \sum_{j=1}^{n} a_{nq}^{j} Q_j(z, w) Q_n(z, w)^{n-j-1} \{\tilde{Q}_n(z, w) w'\}^j = Q_0(z, w) Q_n(z, w)^{n-1},$$

where $\tilde{Q}_n(z, w) = Q_n(z, w)/a_{nq}$. We put for $w = w(z)$

$$U(z) = w^{q+1}/(q+1) + \sum_{k=0}^{q-1} (a_{nk}/a_{nq}) w^{k+1}/(k+1)$$

and

$$V(z) = \sum_{k=0}^{q-1} (a_{nk}/a_{nq})' w^{k+1}/(k+1).$$

Then

$$(4) \qquad \tilde{Q}_n(z, w) w' = U'(z) - V(z)$$

and the poles of $U(z)$ are contained in S. Further, the poles of $U'(z)$ are also contained in S. In fact, substituting (4) into (3), we have

$$(5) \qquad \sum_{j=1}^{n} a_{nq}^{j} Q_j(z, w) Q_n(z, w)^{n-j-1} \{U'(z) - V(z)\}^j = Q_0(z, w) Q_n(z, w)^{n-1}$$

and suppose that $U'(z)$ has a pole at $z = c'$ outside S. The left-hand side of (5) has then a pole at $z = c'$, but the right-hand side of (5) has no pole at $z = c'$, which is a contradiction.

Applying Lemma 1 to U, we have

$$(6) \qquad M(r, U) \leqq K_1 + K_2 r M(r, U') \quad (r \notin E).$$

Let z_r be a point such that

$$M(r, U') = |U'(z_r)|, \quad |z_r| = r \quad (r \notin E).$$

Then,

$$(7) \qquad M(r, U') - M(r, V) \leqq |U'(z_r) - V(z_r)|.$$

Since

$$(8)\qquad M(r, U) \geqq M(r, w)^{q+1}/(q+1) - KM(r, w)^q \left\{\sum_{k=0}^{q-1} M(r, a_{nk})\right\} \Big/ r^d$$

and

$$(9)\qquad M(r, V) \leqq KM(r, w)^q \sum_{k=0}^{q-1} \{rM(r, a_{nk})^2 + M(r, a_{nk})\}/r^{d+1} \quad (r \notin E)$$

by using Lemma 2 if necessary, where $d = \deg a_{nq}$, we have from (6)

$$(10)\qquad M(r, U') - M(r, V) \geqq \{M(r, U) - K_1\}/(K_2 r) - M(r, V)$$
$$\geqq \frac{K}{r} M(r, w)^{q+1} - KM(r, w)^q \left[\sum_{k=0}^{q-1} \{rM(r, a_{nk})^2 + M(r, a_{nk})\} + r^d\right] \Big/ r^{d+1}.$$

Further we have for $j = 0, \cdots, n-1$

$$(11)\qquad |a_{nq}^j(z_r) Q_j(z_r, w) Q_n(z_r, w)^{n-j-1}/a_{nq}^n(z_r)|$$
$$\leqq KM(r, w)^{q_j + q(n-j-1)} \left\{\sum_{k=0}^{q_j} M(r, a_{jk})\right\} \left\{\sum_{k=0}^{q} M(r, a_{nk})\right\} \Big/ r^{d(n-j)}.$$

Applying Lemma 3 to (5) at $z = z_r$ and using (7), (10) and (11), we have

$$M(r, w)^{\min[1, (q+n-p)/n]} \leqq K \left[\max_{0 \leqq j \leqq n-1} \left(\left\{\sum_{k=0}^{q_j} M(r, a_{jk})\right\} \left\{\sum_{k=0}^{q} M(r, a_{nk})\right\} \right)^{1/(n-j)} \right.$$
$$\left. + \sum_{k=0}^{q-1} \{rM(r, a_{nk})^2 + M(r, a_{nk})\} + r^d \right] \Big/ r^d \quad (r \notin E)$$

which reduces to our inequality to be proved by calculating $\log^+$ of the both sides of this inequality.

Theorem 3. *Suppose that all a_{jk} are polynomials and that $p < q+n$. Then, any algebroid solution $w = w(z)$ of the D.E. (1) is algebraic.*

Proof. By Theorems 1 and 2 we obtain

$$T(r, w) = O(\log r) \quad (r \notin E),$$

which shows that w is algebraic.

4. Application. As a special case of Theorem 3, we have the following:

Corollary. *Suppose that all a_{jk} are polynomials. Then, if $q_n = 0$ and $q_j \leqq n-j-1$ $(j = 0, 1, \cdots, n-1)$, any meromorphic solution of the D.E.* (1) *is rational.*

This is an improvement of corollary in [8] and considering the following Eremenko's result ([2]):

"*Suppose that the D.E.* (1) *has an admissible meromorphic solution. Then,*

$$q_j \leqq 2(n-j) \quad (j = 0, 1, \cdots, n)",$$

this is also a generalization of the case of binomial differential equations (see [1], Lemma 1-(i)) to our case.

5. Conjecture. As in [9], we can give the following conjecture:

Conjecture. *When $p < q+n$, any algebroid solution of the D.E.* (1) *would not be admissible.*

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