24. *Notes on Certain Analytic Functions*

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1. Introduction. Let $\mathcal{A}(n)$ denote the class of functions of the form

$$f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathcal{N}=\{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z : |z|<1\}$.

A function $f(z)$ belonging to the class $\mathcal{A}(1)$ is said to be starlike with respect to the origin if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \qquad (z \in \mathcal{U}), \tag{1.2}$$

which is equivalent to

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\pi}{2} \qquad (z \in \mathcal{U}). \tag{1.3}$$

Let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}(1)$ consisting of functions which satisfy

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\pi}{2}\alpha \tag{1.4}$$

for some α $(0<\alpha\leqq 1)$ and for all $z \in \mathcal{U}$. Clearly, a function $f(z)$ belonging to the class $\mathcal{S}^*(\alpha)$ is starlike with respect to the origin in the unit disk $\mathcal{U}$.

Further, a function $f(z)$ in the class $\mathcal{A}(1)$ is said to be convex of order α if it satisfies

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \tag{1.5}$$

for some α $(0\leqq\alpha<1)$ and for all $z \in \mathcal{U}$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}(1)$ consisting of all such functions.

2. Some properties. We begin with the statement of the following lemma due to Miller and Mocanu [1].

Lemma 1. *Let* $f(z)=a+a_n z^n+a_{n+1}z^{n+1}+\cdots$ $(n \in \mathcal{N})$ *be analytic in* $\mathcal{U}$ *with* $f(z)\not\equiv a$. *If* $z_0=r_0 e^{i\theta_0}$ $(0<r_0<1)$ *and*

$$|f(z_0)|=\max_{|z|\leqq r_0}|f(z)|,$$

then

$$\frac{z_0 f'(z_0)}{f(z_0)}=m \tag{2.1}$$

and

$$\mathrm{Re}\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}\right\}\geqq m, \tag{2.2}$$

where $m\geqq 1$ *and*

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(2.3)
$$m \geqq n \frac{|f(z_0)-a|^2}{|f(z_0)|^2-|a|^2} \geqq n \frac{|f(z_0)|-|a|}{|f(z_0)|+|a|}.$$

Applying the above lemma, we derive

Theorem 1. *Let a function $f(z)$ be in the class $\mathcal{A}(n)$ with $f(z) \neq 0$ for $0<|z|<1$. If $z_0=r_0 e^{i\theta_0}$ $(0<r_0<1)$ and*

$$|f(z_0)|=\min_{|z| \leqq r_0} |f(z)|,$$

then

(2.4)
$$\frac{z_0 f'(z_0)}{f(z_0)}=1-m \leqq 0$$

and

(2.5)
$$\mathrm{Re}\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}\right\} \geqq 1-m,$$

where $m \geqq 1$ and

(2.6)
$$m \geqq n \frac{|z_0-f(z_0)|^2}{r_0^2-|f(z_0)|^2} \geqq n \frac{r_0-|f(z_0)|}{r_0+|f(z_0)|}.$$

Proof. We define the function $g(z)$ by

(2.7)
$$g(z)=\frac{z}{f(z)}.$$

Then, by the assumption and the maximum principle, $g(z)$ is analytic in $\mathcal{U}$, $g(0)=1$ and $|g(z)|$ takes its maximum value at $z=z_0=r_0 e^{i\theta_0}$ in the closed disk $|z| \leqq r_0$. It follows from this that

(2.8)
$$|g(z_0)|=\max_{|z| \leqq r_0} |g(z)|=\frac{|z_0|}{|f(z_0)|}.$$

Therefore, applying Lemma 1 to $g(z)$, we observe that

(2.9)
$$\frac{z_0 g'(z_0)}{g(z_0)}=1-\frac{z_0 f'(z_0)}{f(z_0)}=m$$

which shows (2.4) and

(2.10)
$$\begin{aligned}
&\mathrm{Re}\left\{1+\frac{z_0 g''(z_0)}{g'(z_0)}\right\} \\
&=1-\mathrm{Re}\left\{\frac{z_0^2 f''(z_0)}{f(z_0)-z_0 f'(z_0)}\right\}-2\,\mathrm{Re}\left\{\frac{z_0 f'(z_0)}{f(z_0)}\right\} \\
&=1-\left(\frac{1-m}{m}\right)\mathrm{Re}\left\{\frac{z_0 f''(z_0)}{f'(z_0)}\right\}-2(1-m) \\
&\geqq m
\end{aligned}$$

which implies (2.5), where $m \geqq 1$ and

$$m \geqq n \frac{|g(z_0)-1|^2}{|g(z_0)|^2-1}=n \frac{|z_0-f(z_0)|^2}{r_0^2-|f(z_0)|^2} \geqq n \frac{r_0-|f(z_0)|}{r_0+|f(z_0)|}.$$

This completes the assertion of Theorem 1.

Noting that if $f(z) \in \mathcal{A}(n)$ is univalent in $\mathcal{U}$, then $f(z) \neq 0$ for $0<|z|<1$, we have

Corollary 1. *Let a function $f(z)$ in the class $\mathcal{A}(n)$ be analytic and* *univalent in* $\mathcal{U}$. *If* $z_0=r_0e^{i\theta_0}$ $(0<r_0<1)$ *and*

$$|f(z_0)|=\min_{|z|\leqq r_0}|f(z)|,$$

then

$$\frac{z_0f'(z_0)}{f(z_0)}=1-m\leqq 0$$

and

$$\mathrm{Re}\left\{1+\frac{z_0f''(z_0)}{f'(z_0)}\right\}\geqq 1-m,$$

where $m\geqq 1$ *and*

$$m\geqq n\frac{|z_0-f(z_0)|^2}{r_0^2-|f(z_0)|^2}\geqq n\frac{r_0-|f(z_0)|}{r_0+|f(z_0)|}.$$

In order to show next property, we have to recall here the following lemma due to Sheil-Small [3].

Lemma 2. *Let* $f(z)\in\mathcal{A}(1)$ *be starlike with respect to the origin,* $C(r,\theta)=\{f(te^{i\theta}): 0\leqq t\leqq r\}$, *and* $T(r,\theta)$ *be the total variation of* $\arg\{f(te^{i\theta})\}$ *on* $C(r,\theta)$, *so that*

$$T(r,\theta)=\int_0^r\left|\frac{\partial}{\partial t}\arg\{f(te^{i\theta})\}\right|dt. \tag{2.11}$$

Then we have

$$T(r,\theta)<\pi.$$

With the aid of Lemma 2, we prove

Theorem 2. *If* $f(z)\in\mathcal{A}(1)$ *belongs to the class* $\mathcal{K}(\alpha)$ *with* $(1/2)\leqq\alpha<1$, *then* $f(z)\in\mathcal{S}^*(2(1-\alpha))$, *or* $\mathcal{K}(\alpha)\subseteq\mathcal{S}^*(2(1-\alpha))$ *for* $(1/2)\leqq\alpha<1$.

Proof. For a function $f(z)$ belonging to the class $\mathcal{K}(\alpha)$ $((1/2)\leqq\alpha<1)$ we define the function $g(z)$ by

$$1+\frac{zf''(z)}{f'(z)}=\alpha+(1-\alpha)\frac{zg'(z)}{g(z)}. \tag{2.12}$$

Then we see that $g(z)$ is starlike with respect to the origin in $\mathcal{U}$. With an easy calculation, (2.12) leads to

$$\frac{zf'(z)}{f(z)}=\left\{\int_0^z\left(\frac{z}{\zeta}\right)^{1-\alpha}\left(\frac{g(\zeta)}{g(z)}\right)^{1-\alpha}\frac{d\zeta}{z}\right\}^{-1}, \tag{2.13}$$

where the integration in (2.13) is taken along the straight line segment from 0 to z. It follows from (2.13) that

$$\frac{zf'(z)}{f(z)}=\left\{\int_0^1 t^{\alpha-1}\left(\frac{g(tz)}{g(z)}\right)^{1-\alpha}dt\right\}^{-1}. \tag{2.14}$$

An application of Lemma 2 implies that

$$\left|\arg\left(\frac{g(tz)}{g(z)}\right)\right|<\pi \qquad (z\in\mathcal{U}), \tag{2.15}$$

where $0\leqq t\leqq 1$. Letting

$$s=t^{\alpha-1}\left(\frac{g(tz)}{g(z)}\right)^{1-\alpha}, \tag{2.16}$$

(2.14) implies that

$$\arg\left(\frac{zf'(z)}{f(z)}\right)=-\arg\left(\int_0^1 s\,dt\right). \tag{2.17}$$

Since, from (2.15) and (2.16),

$$|\arg(s)|<\pi(1-\alpha), \tag{2.18}$$

we have

$$\left|\arg\left(\int_0^1 s\,dt\right)\right|<\pi(1-\alpha) \qquad (z\in\mathcal{U}) \tag{2.19}$$

by the property of the integral mean (see e.g., [2, Lemma 1]). This proves that

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\pi(1-\alpha) \qquad (z\in\mathcal{U}), \tag{2.20}$$

that is, that $f(z)\in\mathcal{S}^*(2(1-\alpha))$.

Taking $\alpha=1/2$ in Theorem 2, we have

Corollary 2. *If $f(z)\in\mathcal{A}(1)$ belongs to the class $\mathcal{K}(1/2)$, then $f(z)\in\mathcal{S}^*(1)$, or*

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\pi}{2} \qquad (z\in\mathcal{U}).$$

References

[1] S. S. Miller and P. T. Mocanu: Second order differential inequalities in the complex plane. J. Math. Anal. Appl., **65**, 289–305 (1978).

[2] Ch. Pommerenke: On close-to-convex analytic functions. Trans. Amer. Math. Soc., **114**, 176–186 (1965).

[3] T. Sheil-Small: Some conformal mapping inequalities for starlike and convex functions. J. London Math. Soc., **1**, 577–587 (1969).