20. Strong Continuity of the Solution to the Ljapunov Equation $XL-BX=C$ Relative to an Elliptic Operator L

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§1. Introduction. An operator equation, the so called Ljapunov equation, often appears in stabilization studies of linear parabolic systems. The equation is written as $XL-BX=C$, where the operators L, B, and C are given linear operators acting in separable Hilbert spaces, and are derived from a specific boundary feedback control system [6, 7, 8]. A general stabilization scheme for an unstable parabolic equation has been established in [6]. The parabolic equation containing L as a coefficient operator is often affected by small perturbations which may be sometimes interpreted as errors in mathematical formulation of a physical system. In such a case, does the feedback scheme still work for stabilization of the perturbed equation? A study of continuity of a solution X relative to L is fundamental to answer the question. It is the purpose of the paper to examine the continuity of X. We will see in §2 below an affirmative result on this problem.

Let us specify the operators L, B, and C. $\mathcal{L}$ will denote a strongly elliptic differential operator of order 2 in a connected bounded domain Ω of $\mathbb{R}^m$ with a finite number of smooth boundaries Γ of $(m-1)$-dimension;

$$\mathcal{L}u=-\sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^{m}b_i(x)\frac{\partial u}{\partial x_i}+c(x)u,$$

where $a_{ij}(x)=a_{ji}(x)$, $1\leq i,\ j\leq m$, and for some positive δ

$$\sum_{i,j=1}^{m}a_{ij}(x)\xi_i\xi_j\geq\delta|\xi|^2,\quad \xi=(\xi_1,\cdots,\xi_m),\quad x\in\Omega.$$

Associated with $\mathcal{L}$ is a generalized Neumann boundary operator τ;

$$\tau u=\frac{\partial u}{\partial\nu}+\sigma(\xi)u,$$

where $\partial/\partial\nu=\sum_{i,j=1}^{m}a_{ij}(\xi)\nu_i(\xi)\partial/\partial x_j$, and $(\nu_1(\xi),\cdots,\nu_m(\xi))$ indicates the outward normal at $\xi\in\Gamma$. Then, L is defined in $L^2(\Omega)$ by

$$Lu=\mathcal{L}u,\qquad u\in\mathcal{D}(L)=\{u\in H^2(\Omega)\,;\ \tau u=0 \text{ on } \Gamma\}.$$

All norms hereafter will be either $L^2(\Omega)$- or $\mathcal{L}(L^2(\Omega))$-norm unless otherwise indicated. As is well known [2], the spectrum $\sigma(L)$ lies in the interior of a parabola $\{\lambda=\sigma+i\tau\,;\ \sigma=a\tau^2-b,\ \tau\in\mathbb{R}^1\}$, $a>0$. Second, the general structure of the operator B is specified in the following lemma:

Lemma 1.1 [6]. *Let A be a positive-definite self-adjoint operator in a separable Hilbert space H_0 with a compact resolvent. Let $\{\mu_i^2, \zeta_{ij}\,;\ i\geq 1,\ 1\leq j\leq n_i\ (<\infty)\}$ denote the eigenpairs of A (μ_i^2 are labelled according to increasing order, and ζ_{ij} normalized). Define H and B as*

$$H=\mathcal{D}(A^{1/2})\times H_0,$$

and

$$B=\begin{bmatrix}0 & -1\\ A & 2aA^{1/2}\end{bmatrix},\quad \mathcal{D}(B)=\mathcal{D}(A)\times\mathcal{D}(A^{1/2}),\quad a\in(0,1)$$

respectively. Furthermore, set

$$\eta_{ij}^{\pm}=\frac{1}{\sqrt{2}\,\mu_i}\begin{bmatrix}\zeta_{ij}\\ -\mu_i\omega^{\pm}\zeta_{ij}\end{bmatrix},\quad i\geq 1,\ 1\leq j\leq n_i,\quad \omega^{\pm}=a\pm\sqrt{1-a^2}i.$$

Then

(i) $\sigma(B)=\{\mu_i\omega^{\pm};\ i\geq 1\}$, $0<\mu_1<\mu_2<\cdots\to\infty$;

(ii) $B\eta_{ij}^{\pm}=\mu_i\omega^{\pm}\eta_{ij}^{\pm}$, $i\geq 1$, $1\leq j\leq n_i$; *and*

(iii) *the set* $\{\eta_{ij}^{\pm};\ i\geq 1,\ 1\leq j\leq n_i\}$ *forms a normalized Riesz basis for* H.

Remark. Define a real Hilbert space $\hat{H}$ by

$$\hat{H}=\left\{h\in H;\ h=\sum_{i=1}^{\infty}\sum_{j=1}^{n_i}(h_{ij}\eta_{ij}^{+}+\overline{h_{ij}}\eta_{ij}^{-}),\ \sum_{i,j}|h_{ij}|^2<\infty\right\}.$$

Then, it is easy to see that B mapps $\mathcal{D}(B)\cap\hat{H}$ onto $\hat{H}$.

Let c be a positive constant, and set $L_c=L+c$ so that $\sigma(L_c)$ is entirely contained in the right half-plane. Choose real-valued $w_k\in L^2(\Gamma)$, and $\xi_k\in\hat{H}$, $1\leq k\leq N$, N being some integer. Then, the operator C is defined as

$$Cu=-\sum_{k=1}^{N}\langle L_c^{\alpha/2}u, w_k\rangle_{\Gamma}\xi_k,\quad \alpha=\frac{1}{2}+2\varepsilon,\quad 0<\varepsilon<\frac{1}{4},$$

where $\langle\cdot,\cdot\rangle_{\Gamma}$ indicates the inner product in $L^2(\Gamma)$. Physically, w_k's are interpreted as weighting functions for observations located on Γ, and ξ_k's as actuators of a so called compensator [6, 7, 8] in a feedback control system. The number N plays an important role in stabilization studies. Let ξ_k be expressed by $\sum_{i,j}(\xi_{ij}^k\eta_{ij}^{+}+\overline{\xi_{ij}^k}\eta_{ij}^{-})$. Finally, we assume that

$$\sigma(L)\cap\sigma(B)=\emptyset.$$

Under this assumption, we have

Theorem 1.2 [6]. *The Ljapunov equation* $XL-BX=C$ *on* $\mathcal{D}(L)$ *has a unique solution* $X\in\mathcal{L}(L^2(\Omega);H)\cap\mathcal{L}(L_R^2(\Omega);\hat{H})$*). *The solution* X *is expressed by*

$$\text{(1)}\qquad Xu=\sum_{i,j}\sum_{k=1}^{N}f_k(\mu_i\omega^{+};u)\xi_{ij}^k\eta_{ij}^{+}+\sum_{i,j}\sum_{k=1}^{N}f_k(\mu_i\omega^{-};u)\overline{\xi_{ij}^k}\eta_{ij}^{-},$$

$$f_k(\lambda;u)=\langle L_c^{\alpha/2}(\lambda-L)^{-1}u, w_k\rangle_{\Gamma},\qquad 1\leq k\leq N.$$

When $\mathcal{L}$ and τ are perturbed, the resultant operators will be written as

$$\tilde{\mathcal{L}}u=-\sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(\tilde{a}_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^{m}\tilde{b}_i(x)\frac{\partial u}{\partial x_i}+\tilde{c}(x)u,$$

and

$$\tilde{\tau}u=\frac{\partial u}{\partial\tilde{\nu}}+\tilde{\sigma}(\xi)u=\sum_{i,j=1}^{m}\tilde{a}_{ij}(\xi)\nu_i(\xi)\frac{\partial u}{\partial x_j}+\tilde{\sigma}(\xi)u$$

respectively. Then, $\tilde{L}$ is defined by

$$\tilde{L}u=\tilde{\mathcal{L}}u,\qquad u\in\mathcal{D}(\tilde{L})=\{u\in H^2(\Omega);\ \tilde{\tau}u=0\text{ on }\Gamma\}.$$

Here, the symmetry of $\tilde{a}_{ij}$ is not generally assumed, i.e., $\tilde{a}_{ij}\neq\tilde{a}_{ji}$. The

*) $L_R^2(\Omega)$ indicates a subspace of $L^2(\Omega)$ consisting of real-valued functions.

strong ellipticity of $\tilde{L}$ is ensured when $\tilde{a}_{ij}-a_{ij}$ are small enough. $\tilde{X}$ will denote the solution to the Ljapunov equation with L replaced by $\tilde{L}$, i.e., $\tilde{X}\tilde{L}-B\tilde{X}=\tilde{C}=-\sum_{k=1}^{N}\langle\tilde{L}_c^{\alpha/2}\cdot, w_k\rangle_\Gamma\xi_k$. Our goal is to show strong continuity of $\tilde{X}$ relative to $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}$, and $\tilde{\sigma}$. For a control theoretic and geometric property of X, we refer the reader to [6, 7, 8].

§ 2. **Main result.** In order to ensure strong continuity of $\tilde{X}$, we assume throughout the section that $\tilde{a}_{ij}(x)$ and $\tilde{b}_i(x)$, $1\le i, j\le m$, are uniformly bounded in $C^2(\bar{\Omega})$ and so is $\tilde{\sigma}(\xi)$ in $C^2(\Gamma)$. We further assume that ξ_k satisfy

$$\sum_{i,j}\mu_i^{2\alpha}|\xi_{ij}^k|^2<\infty, \qquad 1\le k\le N.$$

Then, our main result is stated as follows:

Theorem 2.1. *The operator $\tilde{X}$ strongly converges to X uniformly in every bounded set of $L^2(\Omega)$ if $\delta=\sum_{i,j=1}^m\|\tilde{a}_{ij}-a_{ij}\|_{C^1(\bar{\Omega})}+\sum_{i=1}^m\|\tilde{b}_i-b_i\|_{C^0(\bar{\Omega})}+\|\tilde{c}-c\|_{C^0(\bar{\Omega})}+\|\tilde{\sigma}-\sigma\|_{C^1(\Gamma)}$ tends to 0.*

Outline of the proof. The operator $\tilde{X}$ is written by (1) with $f_k(\lambda;u)$ replaced by $\tilde{f}_k(\lambda;u)=\langle\tilde{L}_c^{\alpha/2}(\lambda-\tilde{L})^{-1}u, w_k\rangle_\Gamma$. We have to estimate the $L^2(\Gamma)$-norm of

$$h(\lambda)=\tilde{L}_c^{\alpha/2}(\lambda-\tilde{L})^{-1}u-L_c^{\alpha/2}(\lambda-L)^{-1}u, \qquad \lambda=\mu_i\omega^{\pm}.$$

Define an auxiliary operator $\hat{L}$ by

$$\hat{L}u=\tilde{\mathcal{L}}u, \qquad u\in\mathcal{D}(\hat{L})=\mathcal{D}(L).$$

Note that $\hat{L}_c=\hat{L}+c$ is not necessarily an accretive operator. There is a sector $\bar{\Sigma}=\{\lambda=\mu-d;\ \theta\le|\arg\mu|\le\pi\}$, $d>0$, $0<\theta<\pi/2$, such that the resolvents $(\lambda-L)^{-1}$, $(\lambda-\tilde{L})^{-1}$, and $(\lambda-\hat{L})^{-1}$ exist in $\bar{\Sigma}$ and satisfy

$$\|(\lambda-L)^{-1}\|,\quad \|(\lambda-\tilde{L})^{-1}\|,\quad \|(\lambda-\hat{L})^{-1}\|\le\frac{\text{const}}{1+|\lambda|},\quad \lambda\in\bar{\Sigma},$$

and that $\mu_i\omega^{\pm}\in\bar{\Sigma}$, $i\ge 1$. Here, the above constant is independent of δ, and so will be constants appearing below. As is well known [1], $\mathcal{D}(L_c^\gamma)=\mathcal{D}(\tilde{L}_c^\gamma)=H^{2\gamma}(\Omega)$ if $0\le\gamma<3/4$ (constants for the equivalence relations depend on δ). A further analysis via $\hat{L}$ shows

Lemma 2.2. *If $0\le\gamma<3/4$, $\|\tilde{L}_c^\gamma L_c^{-\gamma}\|$ is uniformly bounded, and $\tilde{L}_c^\gamma L_c^{-\gamma}$ strongly converges to 1 as $\delta\to 0$.*

According to m-accretiveness of $\tilde{L}_c^{1/2}$ and $L_c^{1/2}$, we can show

Lemma 2.3. *If $0\le\gamma\le 1/2$, $\|L_c^\gamma\tilde{L}_c^{-\gamma}\|$ is uniformly bounded. As a consequence of Lemma 2.2, $L_c^\gamma\tilde{L}_c^{-\gamma}$ strongly converges to 1 as $\delta\to 0$.*

Given a $g\in H^{1/2}(\Gamma)$, let us consider the boundary value problem

(2) $$(\lambda-\tilde{\mathcal{L}})u=0, \qquad \tilde{\tau}u=g.$$

Lemma 2.4. *There exists a unique solution $u\in H^2(\Omega)$ to eqn. (2) for $\lambda\in\bar{\Sigma}$. The solution u is denoted by $\tilde{N}(\lambda)g$. Then, $\tilde{N}(\lambda)$ belongs to $\mathcal{L}(H^{1/2}(\Gamma);H^2(\Omega))$, and satisfies an estimate*

$$\|\tilde{L}_c^\gamma\tilde{N}(\lambda)g\|\le\text{const}\,|\lambda|^\gamma\|g\|_{H^{1/2}(\Gamma)},\quad \lambda\in\bar{\Sigma},\quad 0\le\gamma<\frac{3}{4}.$$

Before estimating $h(\lambda)$, let us note a relation

$$h(\lambda)=-\tilde{L}_c^{\alpha/2}\tilde{N}(\lambda)(\tilde{\tau}-\tau)(\lambda-\hat{L})^{-1}u+\tilde{L}_c^{\alpha/2}(\lambda-L)^{-1}(\hat{L}-L)(\lambda-\hat{L})^{-1}u$$
$$+(\tilde{L}_c^{\alpha/2}-L_c^{\alpha/2})(\lambda-L)^{-1}u.$$

Based on the preceding lemmas and the trace theorem [2, 5], we estimate $h(\lambda)$ as

$$\|h(\mu_i\omega^{\pm})\|_{L^2(\Gamma)} \leq \text{const}\ \mu_i^{\alpha}\delta\|u\| + \text{const}\ \|(\tilde{L}_c^{\alpha}L_c^{-\alpha} - \tilde{L}_c^{\alpha/2}L_c^{-\alpha/2})L_c^{\alpha}(\mu_i\omega^{\pm} - L)^{-1}u\|.$$

By recalling that each $L_c^{\alpha}(\mu_i\omega^{\pm}-L)^{-1}$ is a compact operator, the second term of the above right-hand side converges to 0 uniformly in $i\geq 1$ and in u (in a bounded set of $L^2(\Omega)$). Thus, the assertion of Theorem 2.1 immediately follows. Details of the proof will appear elsewhere.

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