19. A Theory of Infinite Dimensional Cycles for Dirac Operators

By Hiroshi MORIMOTO
Department of Mathematics, Nagoya University

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1. Introduction. We begin with a general theory and apply it to Dirac operators in the last section. Let $\mathcal{F}=\{F_x\}_{x\in X}$ be a family of Fredholm operators parametrized by an infinite dimensional space X. We are interested in a family (not necessarily a bundle) of solutions of this family of operators.

The family of solutions of operators gives rise to an infinite dimensional cycle κ (called *kernel cycle*) which represents a global structure of the family of solutions. We shall estimate this cycle from below by another cycle ψ (called *index cycle*) determined by the index of the family of operators. Using essentially the vanishing theorem of Lichnerowicz [5], we can show this index cycle is non-trivial for Dirac operators. There is a relation between these cycles and a symplectic geometry, which will be mentioned in forthcoming publications.

Our cycles κ and ψ are motivated by the Catastrophe theory developped by R. Thom [8] and E.C. Zeeman [9]. Especially index cycles ψ are closely related to Thom-Boardman singularities (cf. J.M. Boardman [2], F. Ronga [7] and H. Morimoto [6]).

The method to prove the non-triviality of index cycles for Dirac operators is based on the idea of Atiyah-Jones [1]. They proved non-triviality of characteristic cycles χ. We apply their method to index cycles ψ taking into consideration our estimate $\kappa\supset\psi$.

The detailed proofs will be given elsewhere.

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2. General estimates for cycles. Let X be an infinite dimensional paracompact space, and let $\mathcal{F}=\{F_x\}_{x\in X}$ be a continuous family of Fredholm operators $F_x: E\to E'$, $x\in X$, here E and E' are infinite dimensional Hilbert spaces (or more generally Kuiper spaces). First we set,

$$\chi^*{}_{q,p}(\mathcal{F})=\chi_{p,q}(\mathcal{F}^*)=\{x\in X\,;\,\dim(\ker(F_x^*))\geq p\},$$

where p and q are integers with $p-q=k$ and k is the numerical index of $\mathcal{F}$. This cycle was studied in a general situation by U. Koshorke [4] and its non-triviality was shown by Atiyah-Jones [1] for Dirac operators.

We are concerned with important subcycles of $\chi^*_{q,p}$. Take filtrations of the bundle $E\times X$, $\{E_n\}$, $\{E^{\infty-n}\}_{n=1,2,\dots}$ such that $E\times X=E_n\oplus E^{\infty-n}$ for any n.

We define *kernel cycles* of F by

$$\kappa_{p,q}^{r,s}(\mathrm{Ker}\,(\mathcal{F}))=\{x\in\chi_{q,p}^{*}(\mathcal{F})\,;\,\dim(\ker(F_x)\cap E^{\infty-s})\geq\dim(\ker(F_x))-s+r\},$$

where r and s are integers with $1\leq r\leq s\leq p$.

Remark. The kernel cycles $\kappa_{p,q}^{r,s}$ represent a global structure of the family of solutions of $\mathcal{F}=\{F_x\}$ as follows. Suppose for simplicity $\chi_{q+1,p+1}^{*}(\mathcal{F})=\varnothing$. Then the cohomology class associated to $\kappa_{p,q}^{r,s}$ is a Hankel determinant of Chern classes of the bundle consisting the solutions over $\chi_{q,p}^{*}$.

We give the following estimate for kernel cycles.

Theorem 1. *There exists an infinite dimensional cycle* $\psi_{p,q}^{r,s}(\mathrm{Ind}\,(\mathcal{F}))$ *such that*

$$\kappa_{p,q}^{r,s}(\mathrm{Ker}\,(\mathcal{F}))\supset\psi_{p,q}^{r,s}(\mathrm{Ind}\,(\mathcal{F}))$$

and such that the cohomology class represented by $\psi_{p,q}^{r,s}(\mathrm{Ind}\,(\mathcal{F}))$ *is the following polynomial of Chern classes* $C_i=C_i(\mathrm{Ind}\,(\mathcal{F}))$ *of the index of* $\mathcal{F}$:

$$\begin{vmatrix}(-1)^{p}C_{p}, & \cdots, & (-1)^{p+q+1}C_{p+q-1}\\ \vdots & \ddots & \vdots\\ (-1)^{p+q+1}C_{p-q+1}, & \cdots, & (-1)^{p}C_{p}\end{vmatrix},\ \begin{vmatrix}(-1)^{p+r}C_{p+r}, & \cdots, & (-1)^{p+q+s+1}C_{p+q+2r-s-1}\\ \vdots & \ddots & \vdots\\ (-1)^{p+q+s+1}C_{p-q+s+1}, & \cdots, & (-1)^{p+r}C_{p+r}\end{vmatrix}$$

$$\times(-1)^{pq+(p+r)(q-s+r)}.$$

Remark. The index cycle ψ can be regarded as a version of Thom-Bordman singularities $S^{1,1}$ (cf., Thom [8], Zeeman [9] and Bordman [2], Ronga [7] and Morimoto [6]).

We shall give a brief sketch of the proof. A crucial point to find an index cycle is a construction of a *good* new family of operators parametrized by $\chi_{q,p}^{*}(\mathcal{F})$. A view point of the Catastrophe Theory playes an important role at this point. This new family of operators is found by a deformation of a family of operators

$$\chi_{q,p}^{*}(\mathcal{F})\ni y\longmapsto\pi^{\infty-n}\circ F_y,$$

where $\pi^{\infty-n}$ denotes the projection onto $E^{\infty-n}$. This deformation gives rise to a new cycle i.e., index cycle (analogous to the singularities $S^{1,1}$ in terms of Thom-Boardman singularities). Then it is not difficult to see that the cohomology of this new cycle is determined by the index of $\mathcal{F}$ if we compare it with the original family $\mathcal{F}$ as elements of infinite dimensional k-theory over X and over $\chi_{q,p}^{*}(\mathcal{F})$ (see Elworthy-Tromba [3]).

3. Application to a family of Dirac operators. We shall show the non-triviality of the index cycle $\psi_{p-1,q}^{p-1,p-1}(\mathrm{Ind}\,(\mathcal{D}))$ for the family of coupled Dirac operators. We reduce the problem to the vanishing theorem of Lichnerowicz [5] using the technique of increasing the instanton numbers, the idea of which goes back to Atiyah-Jones [1].

Let P denote a principal bundle over S^4 of instanton number k with the structure group $SU(2)$. Let $\mathcal{A}_k$ denote the space of all the connections on P and let $\mathcal{G}_k$ denote the gauge transformation group (assumed to be identity at some fixed point). We denote by $\mathcal{D}=\{\mathcal{D}_A\}_{A\in\mathcal{A}}$ the family of coupled Dirac operators over P. This gives rise an element of infinite dimensional k-theory over $\mathcal{A}_k/\mathcal{G}_k$. We denote again by $\mathcal{D}$ the corresponding family of operators parametrized by $\mathcal{A}_k/\mathcal{G}_k$.

Theorem 2. *Let p be an odd prime and q an integer less than $(p-1)/3$. Then the cohomology class corresponding to the index cycle $\psi_{p-1,q}^{p-1,p-1}(\mathrm{Ind}\,(\mathcal{D}))$ is non-trivial in $H^*(\mathcal{A}_k/\mathcal{G}_k, Z)$.*

In view of Theorem 1, we have:

Theorem 3. *The carrier of the kernel cycle $\kappa_{p-1,q}^{p-1,p-1}(\mathrm{Ker}\,(\mathcal{D}))$ is not an empty subset of $\mathcal{A}_k/\mathcal{G}_k$.*

The proof of Theorem 2 is parallel to Atiyah-Jones [1]. The difference is that they considered cycles $\chi_{q,p}^*$, while we have to consider subcycles $\kappa_{p,q}^{r,s}\subset\chi_{q,p}^*$. As a result, the corresponding polynomial will be a little bit more complicated. But the essence is quite similar. We should first increase the instanton numbers, second restrict to the selfdual (or anti-selfdual) connections, and finally reduce the whole argument to the vanishing theorem of Lichnerowicz [5].

References

[1] M. F. Atiyah and J. D. S. Jones: Topological aspects of Yang-Mills theory. Commun. Math. Phys., **61**, 97–118 (1978).

[2] J. M. Boardman: Singularities of differentiable maps. I. H. E. S. Publ. Math., **33**, 21–57 (1967).

[3] K. D. Elworthy and A. Tromba: Differential structures and Fredholm maps on Banach manifolds. Proc. Sympos. Pure Math., **15**, Amer. Math. Soc. (1968).

[4] U. Koshorke: Infinite dimensional K-theory and characteristic classes of Fredholm bundle maps. ibid., **15**, 95–133 (1970).

[5] A. Lichnerowicz: Spineurs harmoniques. C. R. Acad. Sci. Paris, Ser. A-B, **257**, 7–9 (1963).

[6] H. Morimoto: Some Morse theoretic aspects of holomorphic vector bundles. Advanced Studies in Pure Math., **3**, Geometry of Geodesics and related Topics. North-Holland, pp. 283–389 (1984).

[7] F. Ronga: Les classes duales aux singularites d'ordre deux. Comm. Math. Helv., **47**, 15–35 (1972).

[8] R. Thom: Structural stability and morphogenesis. Benjamin, New York (1972).

[9] E. C. Zeeman: Catastrophe Theory. Selected papers 1972–1977, Addison-Wesley Publ. Comp. Inc. (1977).