17. A Note on Capitulation Problem for Number Fields

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Let F be a finite extension of a finite algebraic number field k and let C_k and C_F denote the ideal class groups of k and F respectively. A subgroup A of C_k is said to *capitulates* in F if $A \to 1$ under the natural homomorphism $C_k \to C_F$. The principal ideal theorem of class field theory states that C_k always capitulates in Hilbert's class field K over k. However, as shown in Heider-Schmithals [1], for some k, C_k capitulates already in a proper subfield M of K: $k \subseteq M \subseteq K$, $M \neq K$. In the present note, we shall give further simple examples of such number fields k for which the capitulation of C_k occurs in a proper subfield M of Hilbert's class field K over k*).

1. Let L be a finite abelian (or nilpotent) extension over k. For each prime number p, let L_p denote the maximal p-extension over k contained in L, and let $C_{k,p}$ be the p-class group of k, i.e., the Sylow p-subgroup of C_k. It is then easy to see that C_k capitulates in L if and only if $C_{k,p}$ capitulates in L_p for every prime number p. Applying this for Hilbert's class field K over k, we see that a number field M such as stated in the introduction exists if and only if there is a prime number p such that $C_{k,p}$ capitulates in a proper subfield F of Hilbert's p-class field K_p over k: $k \subseteq F \subseteq K_p$, $F \neq K_p$. In what follows, we shall find k such that the 2-class group $C_{k,2}$ capitulates in a proper subfield of Hilbert's 2-class field K_2 over k.

2. Let p, p_1, p_2 be three distinct prime numbers such that

i) $p \equiv p_1 \equiv p_2 \equiv 1 \bmod 4$, $(p/p_1) = (p/p_2) = -1$, the brackets being Legendre's symbol, and that

ii) the norm of the fundamental unit of the real quadratic field $k' = \boldsymbol{Q}(\sqrt{p_1 p_2})$ is 1.

Let

$$k = \boldsymbol{Q}(\sqrt{p p_1 p_2}).$$

By Iyanaga [3], p. 12, we know for the real quadratic field k that

iii) the 2-class group $C_{k,2}$ is an abelian group of type (2, 2) and that

iv) the norm of the fundamental unit of k is -1.

Since $[K_2 : k] = |C_{k,2}| = 4$ for Hilbert's 2-class field K_2 over k, we see immediately that

$$K_2 = \boldsymbol{Q}(\sqrt{p}, \sqrt{p_1}, \sqrt{p_2}).$$

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*) The author was informed by Prof. S. Iyanaga, that he had been reminded of the problem of finding such number fields k by Dr. Li Delang at Sichuan University, China. For various aspects of capitulation problem in general, see Miyake [4].

Let

$$F = kk' = \boldsymbol{Q}(\sqrt{p}, \sqrt{p_1 p_2}).$$

Then

$$k \subseteq F \subseteq K_2, \qquad [K_2 : F] = [F : k] = 2.$$

Proposition. *For the above k, $C_{k,2}$ capitulates in the proper subfield F of K_2. Consequently, the ideal class group C_k of k capitulates in a proper subfield M of Hilbert's class field K over k: $k \subseteq M \subseteq K$, $M \neq K$.*

Proof. Let E_k, $E_{k'}$, and E_F denote the groups of units in k, k', and F respectively. By ii), iv),

$$N_{k'/Q}(E_{k'}) = \{1\}, \qquad N_{k/Q}(E_k) = \{\pm 1\}.$$

Hence it follows from

$$N_{k/Q}(N_{F/k}(E_F)) = N_{k'/Q}(N_{F/k'}(E_F)) \qquad (= N_{F/Q}(E_F))$$

that

$$N_{F/k}(E_F) \neq E_F, \quad \text{i.e.,} \quad H^2(F/k, E_F) \neq 1$$

for the Galois cohomology group of F/k for E_F. Since F/k is a cyclic extension of degree two, we have

$$|H^1(F/k, E_F)| = 2|H^2(F/k, E_F|$$

for the orders of the cohomology groups. Hence we obtain from the above that

$$|H^1(F/k, E_F)| > 2.$$

On the other hand, since F/k is an unramified 2-extension (see [2]),

$$H^1(F/k, E_F) \simeq \text{Ker}\,(C_{k,2} \longrightarrow C_F).$$

Therefore it follows from $|C_{k,2}| = 4$ that

$$\text{Ker}\,(C_{k,2} \longrightarrow C_F) = C_{k,2},$$

namely, that $C_{k,2}$ capitulates in F. Q.E.D.

3. There are many pairs of prime numbers (p_1, p_2), $p_1 \equiv p_2 \equiv 1 \bmod 4$, satisfying the condition ii) in § 2:

$$(p_1, p_2) = (5, 41), \quad (5, 61), \quad (13, 17), \quad \text{etc.}^{**)}$$

Fix one such pair (p_1, p_2). Then, by Dirichlet's theorem on primes in an arithmetic progression, there exist infinitely many p's satisfying the condition i) in § 2. Hence there are infinitely many real quadratic fields k of the form $\boldsymbol{Q}(\sqrt{pp_1p_2})$ such that C_k capitulates in a proper subfield of Hilbert's class field K over k.

Example. Let

$$(p_1, p_2) = (13, 17), \qquad p_1 p_2 = 221.$$

Since

$$\left(\frac{5}{13}\right) = \left(\frac{5}{17}\right) = -1,$$

any prime number p satisfying

$$p \equiv 5 \bmod 884, \qquad 884 = 4 \cdot 13 \cdot 17,$$

provides us a real quadratic field

$$k = \boldsymbol{Q}(\sqrt{221 \cdot p})$$

**) In fact, it seems likely that there exist infinitely many such pairs (p_1, p_2).

such that C_k capitulates in a proper subfield of K. In particular, for $p=5$,

$$k=\boldsymbol{Q}(\sqrt{1105})$$

has class number 4. Hence, in this case,

$$C_k=C_{k,2}, \qquad K=K_2=\boldsymbol{Q}(\sqrt{5}, \sqrt{13}, \sqrt{17})$$

and C_k capitulates in the proper subfield $F=\boldsymbol{Q}(\sqrt{5}, \sqrt{1105})$ of K. H. Wada kindly informed the author that by computing unit groups explicitly, he found that in the above case, C_k capitulates also in $\boldsymbol{Q}(\sqrt{13}, \sqrt{1105})$ as well as in $\boldsymbol{Q}(\sqrt{17}, \sqrt{1105})$. G. Fujisaki then proved in general that the 2-class group $C_{k,2}$ of $k=\boldsymbol{Q}(\sqrt{221\cdot p})$, $p\equiv 5 \bmod 884$, capitulates in all three quadratic subextensions of Hilbert's 2-class field K_2 over k.

Remark. In general, $C_k \neq C_{k,2}$ for $k=\boldsymbol{Q}(\sqrt{pp_1p_2})$ with (p, p_1, p_2) satisfying i), ii) in § 2. For example, the class number of $\boldsymbol{Q}(\sqrt{53\cdot 5\cdot 41})$ is 12.

References

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[3] S. Iyanaga: Sur les classes d'idéaux dans les corps quadratiques. Actualités Sci. Indust., no. 197, Hermann, Paris (1935).

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