14. Global Behavior of Solutions of Quasilinear Ordinary Differential Systems

By Seiji SAITO
Department of Applied Physics, Faculty of Engineering, Osaka University

(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 13, 1989)

1. Introduction. Various kinds of sufficient conditions for the asymptotic behavior of solutions of the quasilinear ordinary differential system

(N) $$x' = A(t, x)x + F(t, x)$$

are obtained by Kartsatos (see Chapter 8 in [2]), where $A(t, x)$ is a real $n \times n$ matrix continuous on $\boldsymbol{R}^+ \times \boldsymbol{R}^n$, $\boldsymbol{R}^+ = [0, +\infty)$, and $F(t, x)$ is an $\boldsymbol{R}^n$-valued function continuous on $\boldsymbol{R}^+ \times \boldsymbol{R}^n$.

Together with the above system, the following linear system

(L) $$x' = B(t)x$$

is concerned, where $B(t)$ is a real $n \times n$ matrix continuous on $\boldsymbol{R}^+$.

Hypothesis 1. *The zero solution of* (L) *is uniformly asymptotically stable.*

Hypothesis 1 holds if and only if the zero solution of (L) is globally exponential-asymptotically stable (see [5]).

In order to investigate the global behavior of solutions of (N), Schauder's fixed point theorem will be applied under the following hypothesis.

Hypothesis 2. *All the solutions of* (N) *for the initial value problems are uniquely determined.*

Theorem 1, in which sufficient conditions for the globally uniform-asymptotic stability of the zero solution of (N) are given, is a strict extention of the well known result for the case where $A(t, x) \equiv A(t)$ (see Remark). In Theorem 2 the condition on perturbed term which is considered by Lasota and Opial [3] ensures boundedness of all the solutions of (N) and the globally uniform attractivity of the zero solution of (N). Moreover in Theorem 3, sufficient conditions for the globally exponential-asymptotic stability of the zero solution of $x' = A(t, x)x$ are obtained by using Liapunov's second method.

2. Preliminaries. The symbol $\|\cdot\|$ will denote a norm in $\boldsymbol{R}^n$ and the corresponding norm for $n \times n$ matrices. Let $C(\boldsymbol{R}^+)$ be the space of $\boldsymbol{R}^n$-valued functions continuous on $\boldsymbol{R}^+$ with the supremum norm $\|\cdot\|_\infty$.

Lemma 1. *Hypothesis* 1 *holds if and only if there exist* $K \geqq 1$ *and* $\lambda > 0$ *such that*

(1) $$\|X_B(t) X_B^{-1}(\tau)\| \leqq K \exp(-\lambda(t - \tau)) \qquad \textit{for } t \geqq \tau,$$

where X_B is a fundamental matrix of solutions of (L) (see [1]).

Now we assume that the following hypothesis holds.

Hypothesis 3. *There exists a $\delta \geqq 0$ such that*

$$\int_0^{+\infty} \sup_{\|x\| \leqq r} \|A(s, x) - B(s)\| ds \leqq \delta \qquad \text{for } r \geqq 0.$$

By the variation of parameters formula, we have for $r \geqq 0$ and y in $C(\boldsymbol{R}^+)$ such that $\|y\|_\infty \leqq r$

$$X_y(t) = X_B(t) X_B^{-1}(\tau) X_y(\tau) + \int_\tau^t X_B(t) X_B^{-1}(s)\{A(s, y(s)) - B(s)\} X_y(s) ds \qquad \text{for } t \geqq \tau,$$

where X_y is a fundamental matrix of solutions of $x' = A(t, y(t))x$. From (1) it follows that

$$\|X_y(t) X_y^{-1}(\tau)\| \exp(\lambda t) \leqq K \exp(\lambda \tau) + K \int_\tau^t \|A(s, y(s)) - B(s)\| \, \|X_y(s) X_y^{-1}(\tau)\| \exp(\lambda s) ds \qquad \text{for } t \geqq \tau.$$

Thus, applying Gronwall's lemma, we have

$$\|X_y(t) X_y^{-1}(\tau)\| \exp(\lambda t) \leqq K \exp\left(\lambda \tau + K \int_\tau^t \|A(s, y(s)) - B(s)\| ds\right),$$

which yields the following lemma.

Lemma 2. *Suppose that Hypotheses* 1 *and* 3 *hold. Then we have for $r \geqq 0$ and y in $C(\boldsymbol{R}^+)$ such that $\|y\|_\infty \leqq r$*

$$\|X_y(t) X_y^{-1}(\tau)\| \leqq K \exp(K\delta - \lambda(t - \tau)) \qquad \text{for } t \geqq \tau. \tag{2}$$

3. Theorems. The global behavior of solutions of (N) is discussed in Theorems 1 and 2 by using Schauder's fixed point theorem.

Theorem 1. *Suppose that Hypotheses* 1–3 *hold and that there exists a non-negative number $C < 1/\{K \exp(K\delta)\}$ such that*

$$\int_0^{+\infty} \sup_{\|x\| \leqq r} \|F(s, x)\| ds \leqq rC \qquad \text{for } r \geqq 0. \tag{3}$$

Then all the solutions of (N) *are uniformly bounded, and the zero solution of* (N) *is uniformly stable and globally uniformly attractive, that is, the zero solution of* (N) *is globally uniform-asymptotically stable.*

Sketch of the proof of Theorem 1. Let $\alpha > 0$ be given arbitrarily. Choose $\beta \geqq K\alpha \exp(K\delta)/\{1 - K \exp(K\delta)C\}$. For $\tau \geqq 0$, $\xi \in \boldsymbol{R}^n$ such that $\|\xi\| < \alpha$, and $i \in \boldsymbol{N}$, we put

$$m_\beta(i) = \max\{\|A(t, x)\| \beta + \|F(t, x)\| : t \in J_i, \|x\| \leqq \beta\},$$

where $J_i = [\tau, \tau + i]$. We consider the following subset $D_\beta(i)$ in $C(J_i)$, where $C(J_i)$ is the space of continuous functions on J_i with the supremum norm,

$$D_\beta(i) = \{y \in C(J_i) : y \text{ satisfies conditions (4)–(6) below}\}.$$

$$y(\tau) = \xi. \tag{4}$$

$$\|y(t)\| \leqq \beta \qquad \text{for } t \in J_i. \tag{5}$$

$$\|y(t) - y(s)\| \leqq m_\beta(i)|t - s| \qquad \text{for } t, s \in J_i. \tag{6}$$

From (4)–(6), $D_\beta(i)$ is a convex and compact subset in $C(J_i)$.

Consider the initial value problem

$$x' = A(t, y(t))x + F(t, y(t)), \qquad x(\tau) = \xi \tag{N_y}$$

for $y \in D_\beta(i)$. It is easily seen that for $y \in D_\beta(i)$ there exists one and only one solution x_y of (N_y) such that

(7) $$x_y(t)=X_y(t)X_y^{-1}(\tau)\xi+\int_\tau^t X_y(t)X_y^{-1}(s)F(s, y(s))ds \qquad \text{for } t\in J_i.$$

By (2), (3) and (7) we obtain

$$x_y(\tau)=\xi \quad \text{and} \quad \|x_y(t)\|\leqq\beta \qquad \text{for } t\in J_i.$$

It follows that $\|x_y(t)-x_y(s)\|\leqq m_\beta(i)|t-s|$ for $t, s\in J_i$. Thus x_y belongs to $D_\beta(i)$ for $y\in D_\beta(i)$.

We can define an operator $V: D_\beta(i)\to D_\beta(i)$ by $V(y)=x_y$. It can be expressed as follows:

$$[V(y)](t)=X_y(t)X_y^{-1}(\tau)\xi+\int_\tau^t X_y(t)X_y^{-1}(s)F(s, y(s))ds \qquad \text{for } t\in J_i.$$

In a similar way to the proof of an analogous part in [4] we can show that V is continuous on $D_\beta(i)$. By applying Schauder's fixed point theorem, V has at least one fixed point in $D_\beta(i)$. Therefore there exists at least one solution $x(\cdot)$ of (N), which belongs to $D_\beta(i)$. Let $J=[\tau, +\infty)$. We obtain a sequence $\{x_i\}$,

$$\{x_i\in C(J): x_i \text{ satisfies conditions (8)–(9) below, for } i\in N\}.$$

(8) For $t\in J_i$, $x_i(t)$ satisfies (4)–(6) as well as (N).

(9) $$x_i(t)=x_i(\tau+i) \qquad \text{for } t\geqq\tau+i.$$

It is clear that $\{x_i\}$ is uniformly bounded and equicontinuous on any compact interval in J. According to Ascoli-Arzela's theorem some subsequence of $\{x_i\}$ converges uniformly on any compact interval in J, the limit of which is a solution of (N) passing through ξ at τ. From Hypothesis 2 it follows that for $\alpha>0$, there exists a $\beta>0$ such that if $\tau\geqq 0$ and $\|\xi\|<\alpha$, then $\|x(t)\|\leqq\beta$ for $t\geqq\tau$, where $x(\cdot)$ is a unique solution of (N) passing through ξ at τ. This implies that all the solutions of (N) are uniformly bounded.

By a similar argument we can prove the uniform stability. The proof of the globally uniform attractivity will be published later. Q.E.D.

Remark. The above theorem is some extention of the well known result as follows: if $A(t,x)\equiv A(t)$ and $\int_0^{+\infty}\|A(s)-B(s)\|ds<+\infty$ under Hypothesis 1, then the zero solution of $x'=A(t)x$ is uniformly asymptotically stable (see [1]).

In the following theorem we require another condition on the perturbed term considered by Lasota and Opial [3].

Theorem 2. *Suppose that Hypotheses* 1–3 *hold and that*

$$\liminf_{r\to+\infty}\frac{1}{r}\int_0^{+\infty}\sup_{\|x\|\leqq r}\|F(s,x)\|ds=0.$$

Then all solutions of (N) *are uniformly bounded and the zero solution of* (N) *is globally uniformly attractive.*

In a similar manner to the proof of Theorem 1 we can prove the above. The details will be published later.

For the special case where $F(t,x)\equiv 0$ we obtain the following theorem by using Liapunov's direct method.

Theorem 3. *Let $F(t, x) \equiv 0$. Suppose that Hypotheses* 1–3 *hold. Then the zero solution of*

$$x' = A(t, x)x \tag{Q}$$

is globally exponential-asymptotically stable.

Shetch of the proof of Theorem 3. Let $\alpha > 0$ be given arbitrarily. From Theorem 1 (or 2) all the solutions of (Q) are uniformly bounded. It follows that there exists a $\beta > 0$ such that if $\tau \geqq 0$ and $\|\xi\| < \alpha$ then $\|x(t)\| \leqq \beta$ for $t \geqq \tau$, where $x(\cdot)$ is a unique solution of (Q) passing through ξ at τ. Let

$$\eta(t) = \|A(t, x(t)) - B(t)\| \qquad \text{for } t \geqq \tau$$

and

$$U(t, x) = \sup\{\|\phi(t+s\,; t, x)\| \exp(\lambda s) : s \geqq 0\} \qquad \text{for } t \geqq \tau,\ \|x\| \leqq \beta,$$

where $\phi(\cdot\,; t, x)$ is the solution of (L) passing through x at t.

Choose a Liapunov function

$$W(t, x) = U(t, x) \exp\left(-K \int_\tau^t \eta(s)ds\right) \qquad \text{for } t \geqq \tau,\ \|x\| \leqq \beta.$$

By a similar way to the proof of Theorem 24.1 in [5], we have for $t \geqq \tau$, $\|x\| \leqq \beta$ and $\|y\| \leqq \beta$

$$\exp(-K\delta)\|x\| \leqq W(t, x) \leqq K\|x\|$$
$$|W(t, x) - W(t, y)| \leqq K\|x - y\|$$

and

$$W'_{(Q)}(t, x(t)) \leqq -\lambda W(t, x(t)),$$

where

$$W'_{(Q)}(t, x) = \limsup_{h \to 0+} \{W(t+h, x+hA(t, x)x) - W(t, x)\}/h \qquad \text{for } t \geqq \tau,\ \|x\| \leqq \beta.$$

This implies that the conclusion holds. Q.E.D.

Acknowledgement. The author is grateful to Prof. Minoru Yamamoto for his helpful suggestions and constant encouragement to this note.

References

[1] J. K. Hale: Ordinary Differential Equations. Krieger, Malabar (1980).
[2] A. G. Kartsatos: Advanced Ordinary Differential Equations. Mariner, Tampa, Florida (1980).
[3] A. Lasota and Z. Opial: Sur les solutions périodiques des équations différentielles ordinaires. Ann. Polon. Math., **16**, 69–94 (1964).
[4] S. Saito and M. Yamamoto: On the existence of periodic solutions for periodic quasilinear ordinary differential systems. Proc. Japan Acad., **63**A, 62–65 (1987).
[5] T. Yoshizawa: Stability theory by Liapunov's second method. Math. Soc. Japan, Tokyo (1966).