13. A Geometric Study on Systems of First Order Differential Equations

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(Communicated by Kôsaku YOSIDA, M. J. A., Feb. 13, 1989)

1. Introduction. Let $J^1(\boldsymbol{R}^n, \boldsymbol{R}^m)$ be the jet space of 1-jets $j_x^1(f)$ of local maps f of $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$. Let $\{x_1, \cdots, x_n\}$ (resp. $\{u_1, \cdots, u_m\}$) be the canonical coordinate system on $\boldsymbol{R}^n$ (resp. $\boldsymbol{R}^m$). Then we can introduce the coordinate system $\{x_1, \cdots, x_n, u_1, \cdots, u_m, \cdots, p_j^i, \cdots\}$ on $J^1(\boldsymbol{R}^n, \boldsymbol{R}^m)$ associated with $\{x_1, \cdots x_n, u_1, \cdots, u_m\}$ given by $p_j^i = \partial u_i/\partial x_j$. Let π_1 (resp. π_2) be the usual projection of $J^1(\boldsymbol{R}^n, \boldsymbol{R}^m)$ onto $\boldsymbol{R}^n$ (resp. $\boldsymbol{R}^m$). In the following we assume that $n=m=2$ and consider a system of differential equations

$$(E) \qquad \begin{cases} F_1(p) \equiv \alpha_1(x)p_1^1 + \beta_1(x)p_1^2 + \alpha_2(x)p_2^1 + \beta_2(x)p_2^2 = 0, \\ F_2(p) \equiv \gamma_1(x)p_1^1 + \delta_1(x)p_1^2 + \gamma_2(x)p_2^1 + \delta_2(x)p_2^2 = 0 \end{cases}$$

on $J^1(\boldsymbol{R}^2, \boldsymbol{R}^2)$. Denote by $\mathcal{S}(E)$ the set of local solutions of E and set $S(E) = \{j_x^1(f)\,;\, f \in \mathcal{S}(E)$ *and* $x \in$ *the domain of* $f\}$ and $I(E) = \{p \in J^1(\boldsymbol{R}^2, \boldsymbol{R}^2)\,;\, F_1(p) = F_2(p) = 0\}$. Then, in general, we have $I(E) \supset S(E)$.

Let us consider the category $\tilde{C}$ of systems of differential equations E which satisfy the following properties around $p_o \in J^1(\boldsymbol{R}^2, \boldsymbol{R}^2)$:

(1) $I(E) = S(E)$,

(2) $\det \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \neq 0 \quad (i=1, 2)$,

(3) $(\alpha_2\beta_1 - \alpha_1\beta_2)(\gamma_2\delta_1 - \gamma_1\delta_2)(\beta_1\delta_2 - \beta_2\delta_1) \neq 0$.

Denote by $\mathcal{A}(E)$ the pseudogroup of local transformations ϕ on $\boldsymbol{R}^2$ such that, for any $s \in \mathcal{S}(E)$, if $\phi \circ s$ is defined, then $\phi \circ s \in \mathcal{S}(E)$. $\mathcal{A}(E)$ is called the automorphism pseudogroup of E. Then, according to [2], for any element $E \in \tilde{C}$, we have

Proposition 1.1. *The system of defining equations of* $\mathcal{A}(E)$ *around* $x_o = \pi_1(p_o)$ *is given by*

$$\begin{cases} \partial\phi_1/\partial u_1 = a(x)(\partial\phi_1/\partial u_2) + \partial\phi_2/\partial u_2, \\ \partial\phi_2/\partial u_1 = b(x)(\partial\phi_1/\partial u_2) \end{cases}$$

where $\phi = (\phi_1, \phi_2) \in \mathcal{A}(E)$ *and* $a(x) = (\beta_1\delta_2 - \beta_2\delta_1)^{-1}(\beta_1\gamma_2 - \alpha_2\delta_1 + \alpha_1\delta_2 - \beta_2\gamma_1)$, $b(x) = (\beta_1\delta_2 - \beta_2\delta_1)^{-1}(\alpha_2\gamma_1 - \alpha_1\gamma_2)$.

We set $C = \{E \in \tilde{C}\,;\, a(x)$ *and* $b(x)$ *are constant*$\}$. The purpose of this note is to classify systems of differential equations belonging to C from the geometrical viewpoint using the couple of real numbers (a, b) which is called the structure vector of $E \in C$.

2. Preliminary lemma. Let us consider the 4-dimensional Euclidean space $\boldsymbol{R}^4$ with the canonical coordinate system $\{v_1, v_2, v_3, v_4\}$ and a vector field $W = (av_1 + v_2)(\partial/\partial v_1) + bv_1(\partial/\partial v_2) + (av_3 + v_4)(\partial/\partial v_3) + bv_3(\partial/\partial v_4)$ where a and

b are arbitrary real constants. Denote by $\boldsymbol{P}^3$ the 3-dimensional real projective space and let π be the canonical projection of $\boldsymbol{R}^4-\{0\}$ onto $\boldsymbol{P}^3$.

Lemma 2.1. *To the vector field W on $\boldsymbol{R}^4$, there corresponds a vector field X on $\boldsymbol{P}^3$ such that, for any $\tilde{p}\in\boldsymbol{R}^4-\{0\}$, we have $\pi_*(W_{\tilde{p}})=X_p$ where $p=\pi(\tilde{p})$.*

The proof is easily done by using the inhomogeneous coordinate system.

3. Statement of results. For any local transformation ϕ on $\boldsymbol{R}^2$, we define the lift $\phi^{(1)}$ of ϕ to $J^1(\boldsymbol{R}^2,\boldsymbol{R}^2)$ by $\phi^{(1)}(j^1_x(f))=j^1_x(\phi\circ f)$. Then we can define the pseudogroup $\mathcal{A}(E)^{(1)}$ on $J^1(\boldsymbol{R}^2,\boldsymbol{R}^2)$ which is generated by $\{\phi^{(1)};\phi\in\mathcal{A}(E)\}$. Similarly we can define the lift $X^{(1)}$ to $J^1(\boldsymbol{R}^2,\boldsymbol{R}^2)$ of any local vector field X on $\boldsymbol{R}^2$. A vector field X is called an $\mathcal{A}(E)$-vector field if the local 1-parameter group of local transformations ϕ_t generated by X is included in $\mathcal{A}(E)$. Denote by $\mathcal{L}(E)$ the sheaf on $\boldsymbol{R}^2$ of germs of local $\mathcal{A}(E)$-vector fields. Then we can define the sheaf $\mathcal{L}(E)^{(1)}$ on $J^1(\boldsymbol{R}^2,\boldsymbol{R}^2)$ of germs of vector fields $X^{(1)}$ where X is any local cross-section of $\mathcal{L}(E)$.

A function f defined around $p\in J^1(\boldsymbol{R}^2,\boldsymbol{R}^2)$ is called a differential invariant of $\mathcal{A}(E)$ if $Zf=0$ for any $Z\in\mathcal{L}(E)^{(1)}_p$ which is the stalk of $\mathcal{L}(E)^{(1)}$ on p.

Proposition 3.1. *Let $E\in\mathcal{C}$ with the structure vector (a,b). Then a function f given around $p\in J^1(\boldsymbol{R}^2,\boldsymbol{R}^2)$ is a differential invariant of $\mathcal{A}(E)$ if and only if f satisfies the following relations around p:*

$$\begin{aligned}W^E(f)\equiv&(ap^1_1+p^2_1)(\partial f/\partial p^1_1)+bp^1_1(\partial f/\partial p^2_1)\\&+(ap^1_2+p^2_2)(\partial f/\partial p^1_2)+bp^1_2(\partial f/\partial p^2_2)=0,\\Z(f)\equiv&\, p^1_1(\partial f/\partial p^1_1)+p^2_1(\partial f/\partial p^2_1)+p^1_2(\partial f/\partial p^1_2)+p^2_2(\partial f/\partial p^2_2)=0,\\&\partial f/\partial u_1=0,\qquad \partial f/\partial u_2=0.\end{aligned}$$

As for the proof, see [2, Proposition 6.2].

W^E and Z are considered as vector fields on $J^1_{xz}=\{p\in J^1(\boldsymbol{R}^2,\boldsymbol{R}^2);\pi_1(p)=x,\pi_2(p)=z\}\cong\boldsymbol{R}^4$ and we can prove that $\pi_*(Z)=0$ and by Lemma 2.1 we have the vector field $X^E=\pi_*(W^E)$ on $\boldsymbol{P}^3$. X^E is called the characteristic vector field of $E\in\mathcal{C}$.

Proposition 3.2. *Assume that $b\neq 0$. Then X^E admits a singular point if and only if $a^2+4b\geqq 0$.*

This is proved by the local expression of X^E.

Let E be an element in $\mathcal{C}$ with the structure vector (a,b), $b\neq 0$. Denote by P^E the set of nonsingular points of X^E. Then P^E is open and dense in $\boldsymbol{P}^3$. If $a^2+4b<0$, then by Proposition 3.2 we have $P^E=\boldsymbol{P}^3$. The vector field X^E gives a foliation $\mathcal{F}^E$ on P^E such that any leaf of $\mathcal{F}^E$ is an integral curve of X^E ([1]).

Definition 3.1. Let $\mathcal{F}$ be a foliation of codim q on a manifold M given by the following transverse structure $(\{U_\alpha,f_\alpha\},\{\gamma_{\alpha\beta}\},\{\boldsymbol{R}^q_\alpha\})$ where

i) $\{U_\alpha\}$ is an open covering of M,

ii) $f_\alpha: U_\alpha\to\boldsymbol{R}^q_\alpha$ is a submersion,

iii) $f_\alpha=\gamma_{\alpha\beta}\circ f_\beta$ on $U_\alpha\cap U_\beta$ where $\gamma_{\alpha\beta}:\boldsymbol{R}^q_\alpha\to\boldsymbol{R}^q_\beta$ are local diffeomorphisms.

$\mathcal{F}$ is called an algebraic foliation if each $\gamma_{\alpha\beta}$ is a rational map i.e. there exist polynomials $r^i_{\alpha\beta}(x_1, \cdots, x_q)$ $(i=1, \cdots, q)$ and $s^j_{\alpha\beta}(x_1, \cdots, x_q)$ $(j=1, \cdots, q)$ such that $s^j_{\alpha\beta}(x_1, \cdots, x_q) \neq 0$ for any j and any $(x_1, \cdots, x_q) \in f_\beta(U_\alpha \cap U_\beta)$ and that $\gamma^i_{\alpha\beta} = r^i_{\alpha\beta}/s^i_{\alpha\beta}$ where $\gamma_{\alpha\beta}=(\gamma^1_{\alpha\beta}, \cdots, \gamma^q_{\alpha\beta})$.

Let $(a(E), b(E))$ denote the structure vector of $E \in \mathcal{C}$. We set $\mathcal{C}' = \{E \in \mathcal{C}\,;\, b(E) \neq 0\}$, $\mathcal{C}'_+ = \{E \in \mathcal{C}'\,;\, a(E)^2 + 4b(E) > 0\}$, $\mathcal{C}'_0 = \{E \in \mathcal{C}'\,;\, a(E)^2 + 4b(E) = 0\}$ and $\mathcal{C}'_- = \{E \in \mathcal{C}'\,;\, a(E)^2 + 4b(E) < 0\}$. Then our main results are

Theorem 3.3. *Let $E \in \mathcal{C}'$. Then the foliation $\mathcal{F}^E$ on P^E is an algebraic foliation.*

Theorem 3.4. *$E \in \mathcal{C}'$ is elliptic if and only if $E \in \mathcal{C}'_-$.*

Theorem 3.5. *Let E_1 and E_2 be in $\mathcal{C}'$. Then the foliation $\mathcal{F}^{E_1}$ on P^{E_1} is isomorphic to the foliation $\mathcal{F}^{E_2}$ on P^{E_2} if and only if both E_1 and E_2 belong to the same one of the three classes $\mathcal{C}'_+$, $\mathcal{C}'_0$ and $\mathcal{C}'_-$.*

4. Proof of Theorem 3.3. For $\tilde{U}_1 = \{\tilde{p} \in \boldsymbol{R}^4 - \{0\}\,;\, v_1(\tilde{p}) \neq 0\}$, let $\{x, y, z\}$ be the coordinate system on $U_1 = \pi(\tilde{U}_1) \subset \boldsymbol{P}^3$ associated with $\{v_1, v_2, v_3, v_4\}$. We choose a point $p_0 \in U_1$ satisfying $(x^2+ax-b)(p_0) \neq 0$. Then $p_0 \in P^E$ because it is proved that X^E is written on U_1 by $X^E = (-x^2-ax+b)(\partial/\partial x) + (z-xy)(\partial/\partial y) + (-xz-az+by)(\partial/\partial z)$. We set $\bar{U}_1 = \{p \in P^E \cap U_1\,;\, (x^2+ax-b)(p) \neq 0\}$. Then, by setting $\bar{I}^{ab}_1 = (xy-z)/(x^2+ax-b)$ and $\bar{J}^{ab}_1 = (az+xz-by)/(x^2+ax-b)$, the map $\bar{f}_1 : \bar{U}_1 \to \boldsymbol{R}^2$ defined by $\bar{f}_1(p) = (\bar{I}^{ab}_1(p), \bar{J}^{ab}_1(p))$ is a submersion. Note that $\pi^*(\bar{I}^{ab}_1)$ and $\pi^*(\bar{J}^{ab}_1)$ are differential invariants of $\mathcal{A}(E)$. If $p \in \bar{U}_1 \cap P^E$ satisfies $(x^2+ax-b)(p)=0$, then it is proved that $(z-xy)(p) \neq 0$. By setting $\hat{I}^{ab}_1 = (xy-z)/\{(xy-z)+(x^2+ax-b)\}$ and $\hat{J}^{ab}_1 = (az+xz-by)/\{(xy-z)+(x^2+ax-b)\}$, the map $\hat{f}_{1p} : \hat{U}_{1p} \to \boldsymbol{R}^2$ defined on a neighborhood $\hat{U}_{1p}$ of p by $\hat{f}_{1p}(q) = (\hat{I}^{ab}_1(q), \hat{J}^{ab}_1(q))$ is a submersion. Thus we get an open covering $\{\bar{U}_i, \hat{U}_{ip}\,;\, p \in U_i \backslash \bar{U}_i, i=1,2,3,4\}$ of P^E and submersions $\bar{f}_i : \bar{U}_i \to \boldsymbol{R}^2$ and $\hat{f}_{ip} : \hat{U}_{ip} \to \boldsymbol{R}^2$. It is proved that we have

$$\bar{I}^{ab}_3 = \bar{I}^{ab}_1/\{b(\bar{I}^{ab}_1)^2 - a\bar{I}^{ab}_1\bar{J}^{ab}_1 - (\bar{J}^{ab}_1)^2\},$$
$$\bar{J}^{ab}_3 = -(a\bar{I}^{ab}_1 + \bar{J}^{ab}_1)/\{b(\bar{I}^{ab}_1)^2 - a\bar{I}^{ab}_1\bar{J}^{ab}_1 - (\bar{J}^{ab}_1)^2\},$$
$$\bar{I}^{ab}_2 = \bar{I}^{ab}_1, \qquad \bar{J}^{ab}_2 = -a\bar{I}^{ab}_1 - \bar{J}^{ab}_1.$$

By continuing these arguments, it is proved that we get the sets $\{(\bar{U}_i, \bar{f}_i), (\hat{U}_{ip}, \hat{f}_{ip})\,;\, 1 \leqq i \leqq 4,\ p \in U_i \backslash \bar{U}_i\}$ and $\{\gamma_{ij}, \gamma_{ip}, \gamma_{ipjq}\,;\, 1 \leqq i, j \leqq 4,\ p \in U_i \backslash \bar{U}_i,\ q \in U_j \backslash \bar{U}_j\}$ such that $\bar{f}_i = \gamma_{ij} \circ \bar{f}_j$, $\bar{f}_i = \gamma_{ip} \circ \hat{f}_{ip}$ and $\hat{f}_{ip} = \gamma_{ipjq} \circ \hat{f}_{jq}$ where γ_{ij}, γ_{jq} and γ_{ipjq} are rational maps and that they give an algebraic foliation on P^E which is just the foliation $\mathcal{F}^E$. This is the outline of the proof.

5. Proof of Theorem 3.4. E is said to be elliptic if, for any $(t_1, t_2) \in \boldsymbol{R}^2 - \{0\}$, the matrix $t_1M_1 + t_2M_2$, $M_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$, is nonsingular. Since $\det(t_1M_1 + t_2M_2) = (\alpha_1\delta_1 - \beta_1\gamma_1)t_1^2 + (\alpha_1\delta_2 + \alpha_2\delta_1 - \beta_1\gamma_2 - \beta_2\gamma_1)t_1t_2 + (\alpha_2\delta_2 - \beta_2\gamma_2)t_2^2$, we see that E is elliptic if and only if $(\alpha_1\delta_2 + \alpha_2\delta_1 - \beta_1\gamma_2 - \beta_2\gamma_1)^2 - 4(\alpha_1\delta_1 - \beta_1\gamma_1)(\alpha_2\delta_2 - \beta_2\gamma_2) \equiv (\beta_1\delta_2 - \beta_2\delta_1)^2(a^2+4b) < 0$. This proves Theorem 3.4 because $E \in \mathcal{C}' \subset \tilde{\mathcal{C}}$ means $\beta_1\delta_2 - \beta_2\delta_1 \neq 0$.

6. Proof of Theorem 3.5. It is easy to prove that, if $\mathcal{F}^{E_1}$ is isomorphic to $\mathcal{F}^{E_2}$, then both E_1 and E_2 belong to the same class. Conversely we assume that they are in the same class. If we can find a linear transformation ϕ on $\boldsymbol{R}^4$ satisfying $\phi_*(W_1)=\lambda W_2+\sigma Z$ for some real numbers λ and σ, then ϕ induces an isomorphism of $\mathcal{F}^{E_1}$ to $\mathcal{F}^{E_2}$. In particular, it is sufficient to find $\phi=(c^i_j)_{1\leqq i,j\leqq 4}$ such that $c^i_j=0$ for $1\leqq i\leqq 2$, $3\leqq j\leqq 4$ and for $3\leqq i\leqq 4$, $1\leqq j\leqq 2$ and that

$$(6.1)_k \quad \begin{cases} c^k_k a_1+c^k_{k+1}b_1=\lambda(a_2c^k_k+c^{k+1}_k)+\sigma c^k_k, & c^{k+1}_k=\lambda b_2c^k_{k+1}+\sigma c^{k+1}_{k+1}, \\ c^{k+1}_k a_1+c^{k+1}_{k+1}b_1=\lambda b_2c^k_k+\sigma c^{k+1}_k, & c^k_k=\lambda(a_2c^k_{k+1}+c^{k+1}_{k+1})+\sigma c^k_{k+1} \quad (k=1,3) \end{cases}$$

for some real numbers λ and σ.

We choose real numbers α and β satisfying $2\alpha-\beta a_1\neq 0$ and $\alpha^2-a_1\alpha\beta-b_1\beta^2\neq 0$ and set $A_1=2\alpha-\beta a_1$, $A_2=\alpha a_1+2\beta b_1$ and $B=-\alpha^2+a_1\alpha\beta+b_1\beta^2$. Let us consider the algebraic equation with respect to δ

$$(6.2) \qquad (a_1^2+4b_1)A_1^{-2}B\delta^2+(a_1^2+4b_1)a_2\beta A_1^{-2}B\delta+a_2^2A_1^{-2}B^2-b_2B=0.$$

Then, under the condition $a_1^2+4b_1\neq 0$, (6.2) admits a real solution δ if and only if $(a_1^2+4b_1)(a_2^2+4b_2)\geqq 0$. We set $\gamma=A_1^{-1}(\delta A_2+a_2B)$. Then α,β,γ and δ satisfy

$$(6.3) \qquad A_1\gamma-A_2\delta=a_2B, \quad a_1\gamma\delta+b_1\delta^2-\gamma^2=-b_2B, \quad \alpha\delta-\beta\gamma\neq 0.$$

If $a_1^2+4b_1=a_2^2+4b_2=0$, (6.2) holds identically and we can choose γ and δ such that α,β,γ and δ satisfy (6.3).

Now we set $\gamma=-B/(\alpha\delta-\beta\gamma)$, $\sigma=(a_2B+\alpha\delta a_1+\beta\delta b_1-\alpha\gamma)/(\alpha\delta-\beta\gamma)$. Then, by (6.3), we get $(-\gamma^2+\gamma\delta a_1+\delta^2b_1)/(\alpha\delta-\beta\gamma)b_2=\lambda$ and $(\alpha\gamma-\beta\gamma a_1-\beta\delta b_1)/(\alpha\delta-\beta\gamma)=\sigma$. If we set $c^k_k=\alpha$, $c^k_{k+1}=\beta$, $c^{k+1}_k=\gamma$ and $c^{k+1}_{k+1}=\delta$, then λ and σ satisfy $(6.1)_k$. This is the outline of the proof.

References

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