11. A Certain Functional Derivative Equation Corresponding to $\Box u+cu+bu^2+au^3=g$ on R^{d+1}

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Introduction and results. L^p_r $(1\le p\le\infty, r\in \boldsymbol{R})$ denotes the space of weighted p-summable functions on $\boldsymbol{R}^d$ with norm given by $|u|_{p,r}=\left(\int_{\boldsymbol{R}^d}(1+|x|^2)^{rp/2}|u(x)|^p dx\right)^{1/p}$ or $|u|_{\infty,r}=\text{ess. sup}_{x\in\boldsymbol{R}^d}(1+|x|^2)^{r/2}|u(x)|$. When $r=0$, we put $L^p=L^p_0$ with $|u|_p=|u|_{p,0}$. For $s\in \boldsymbol{N}$, $\|u\|_{s,r}=\left(\int_{\boldsymbol{R}^d}(1+|x|^2)^r \sum_{|\alpha|\le s}|D^\alpha u(x)|^2dx\right)^{1/2}$ represents the norm of H^s_r, the weighted Sobolev space of order s on $\boldsymbol{R}^d$. For general $s\in\boldsymbol{R}$, H^s_r is defined by using the interpolation theory and H^s stands for H^s_0 with $\|u\|_s=\|u\|_{s,0}$. The dual space of L^p_r is L^q_{-r} for $1\le p<\infty$ with $1/p+1/q=1$. $H^{-s}_{-r}=(\mathring{H}^s_r)^*$ for $s\ge 0$ with $\mathring{H}^s_r=\mathring{H}^s_r(\boldsymbol{R}^d)$ $(s\ge 0)$ being the closure of $C^\infty_0(\boldsymbol{R}^d)$ in H^s_r.

Now, we put $X={}^t(V\times L^2)$ and $X^*=V^*\times L^2$ with norms $\|U\|_X=\|u\|_V+|v|_2$ and $\|\Xi\|_{X^*}=\|\xi\|_{V^*}+|\eta|_2$ for $U={}^t(u,v)$ and $\Xi=(\xi,\eta)$. Here, $V=H^1\cap L^4$ and $V^*=H^{-1}+L^{4/3}$ with norms $\|u\|_V=\|u\|_1+|u|_4$ and $\|\xi\|_{V^*}=\inf_{\xi=\xi_1+\xi_2}(\|\xi_1\|_{-1}+|\xi_2|_{4/3})$.

Our aim of this paper is to solve the following problems: Let $0<T_0\le\infty$.

(I) Find a functional $W(t,\Xi)$ on $t\in(0,T_0)\times X^*$ satisfying

$$\text{(I.1)}\quad \frac{\partial}{\partial t}W(t,\Xi)=\int_{\boldsymbol{R}^d}\left[\eta(x)\left((\Delta-c)\frac{\delta W(t,\Xi)}{\delta\xi(x)}+ib\frac{\delta^2W(t,\Xi)}{\delta\xi(x)^2}+a\frac{\delta^3W(t,\Xi)}{\delta\xi(x)^3}\right)+\xi(x)\frac{\delta W(t,\Xi)}{\delta\eta(x)}+i\eta(x)g(x,t)W(t,\Xi)\right]dx,$$

$$\text{(I.2)}\qquad W(t,0)=1,\qquad W(0,\Xi)=W_0(\Xi).$$

Here given data are $W_0(\Xi)$ and $g(x,t)$.

(II) Find a family of Borel measures $\{\mu(t,dU)\}_{0<t<T_0}$ on X satisfying

$$\text{(II)}\quad \int_0^{T_0}\int_X\frac{\partial\Phi(t,U)}{\partial t}\mu(t,dU)dt+\int_X\Phi(0,U)\mu_0(dU)$$
$$=-\int_0^{T_0}\int_X\int_{\boldsymbol{R}^d}\left[(\Delta u(x)-f(u(x))+g(x,t))\frac{\delta\Phi(t,U)}{\delta v(x)}+v(x)\frac{\delta\Phi(t,U)}{\delta u(x)}\right]\times dx\mu(t,dU)dt$$

for suitable 'test functionals' $\Phi(t,U)$ with given data $\mu_0(dU)$ and $g(x,t)$.

For the notational simplicity, we put here $f(u)=au^3+bu^2+cu$, $F(u)=au^4/4+bu^3/3+cu^2/2$ and

$$H(U)=H(u,v)=\int_{\boldsymbol{R}^d}\{|v(x)|^2/2+|\nabla u(x)|^2/2+F(u(x))\}dx.$$

Assume that

(AS 0) $\qquad a>0$ and $b^2 \leq \frac{9}{2}ac$ with $\kappa=\frac{a}{4}-\frac{b^2}{18c}\geq 0$.

For $0<\delta<1$ and $0<r$, we define auxiliary function spaces as $\tilde{V}=\mathring{H}^{1-\delta}_{-r} \cap L^3_{-r/3}$, $\tilde{V}^*=H^{-1+\delta}_r+L^{3/2}_{r/3}$, $\tilde{X}={}^t(\tilde{V}\times H^{-\delta}_{-r})$ and $\tilde{X}^*=\tilde{V}^*\times \mathring{H}^{\delta}_r$. Defining a non-negative functional $\Lambda(U)=\|u\|_{1-\delta,-r}+|u|^3_{3,-r/3}+\|v\|_{-\delta,-r}$ on $\tilde{X}$, we introduce the notion of test functionals as follows.

Definition 1. A real function $\Phi(\cdot,\cdot)$ defined on $[0,T_0)\times\tilde{X}$ is called a test functional if it satisfies the following:

(1) $\Phi(\cdot,\cdot)$ is continuous on $[0,T_0)\times\tilde{X}$ and verifies $\sup_{(t,U)}|\Phi_t(t,U)|/(1+\Lambda(U))<\infty$.

(2) $\Phi(\cdot,\cdot)$ is Fréchet $\tilde{X}$-differentiable in the direction X. Moreover, $\Phi_U(\cdot,\cdot)$ is continuous form $[0,T_0)\times X$ to $\tilde{X}^*$ and is bounded, i.e. $\Phi_u(t,U)\in C_b(0,T_0;\tilde{V}^*)$, $\Phi_v(t,U)\in C_b(0,T_0;\mathring{H}^{\delta}_r)$.

(3) There exists $0<T\leq T_0$, $T<\infty$, depending on Φ such that $\Phi(t,U)=0$ for any $t\geq T$ and $U\in\tilde{X}$. (In this case, Φ is said to have the compact support in t.)

Now, we introduce the notion of solutions.

Definition 2. A family of Borel measures $\{\mu(t,dU)\}_{0<t<T_0}$ on X is called a strong solution of Problem (II) on $(0,T_0)$ if it satisfies the following conditions:

(1) $\int_X (1+\Lambda(U))\mu(\cdot,dU)\in L^\infty(0,T_0)$.

(2) $\int_X \Phi(U)\mu(t,dU)$ is measurable in t for any non-negative, weakly continuous functional $\Phi(\cdot)$ on X.

(3) For any test functional $\Phi(\cdot,\cdot)$, it satisfies (II).

Definition 3. A functional $W(t,\Xi)$ defined on $[0,T_0)\times X^*$ will be called a strong solution of problem (I) on $(0,T_0)$ if it satisfies:

(1) For each $\Xi\in\tilde{X}^*$, $W(t,\Xi)$ belongs to $L^1[0,T_0)$ and continuous at $t=0$.

(2) $W(t,\Xi)$ is three times Fréchet X^*-differentiable in the direction $\tilde{X}^*$ for a.e.t. Moreover, $\delta^k W(t,\Xi)/\delta\xi(x)^k$ with $1\leq k\leq 3$ and $\delta W(t,\Xi)/\delta\eta(x)$ exist as elements in $\mathcal{D}'(\boldsymbol{R}^d)$ for a.e.t.

(3) $W(t,\Xi)$ satisfies (I.1)–(I.4) as distributions in t for each $\Xi\in\tilde{X}^*_\infty\equiv \cup_{m=1}^\infty \Pi_m\tilde{X}^*$ (see below).

Our results are

Theorem A. *Put* $E_*(U)=|v|_2^2/2+\max(1/2,\,c/2+|b|/6)\|u\|_1^2+(a/4+|b|/6)|u|_4^4$. *Under Assumption* (AS 0), *for any Borel probability measure* $\mu_0(dU)$ *on* X *satisfying*

(AS 1) $\qquad \int_X (1+E_*(U))^\alpha \mu_0(dU)<\infty \qquad$ *for* $\begin{cases}\alpha=1 & \textit{when } \kappa>0,\\ \alpha>3/2 & \textit{when } \kappa=0,\ d\leq 3\end{cases}$

and any $g\in L^2(0,T_0;L^2)\cap L^\infty(0,T_0;V^*)$, *there exists a solution* $\{\mu(t,dU)\}_{0<t<T_0}$ *of* (II).

Theorem B. *Assume that* (AS 0) *holds. Let a positive definite functional* $W_0(\Xi)$ *on* X^* *be given which is three times Fréchet* X^**-differentiable in the direction* $\tilde{X}^*$ *having* $\delta^k W_0(\Xi)/\delta\xi(x)^k$ *with* $1 \le k \le 3$ *and* $\delta W_0(\Xi)/\delta\eta(x)$ *in* $\mathcal{D}'(\boldsymbol{R}^d)$. *Then, for any* $g \in L^2(0, T_0; L^2) \cap L^\infty(0, T_0; V^*)$, *there exists a strong solution* $W(t, \Xi)$ *of Problem* (I).

Sketch of proofs. For (I) and (II), we may correspond the following nonlinear Klein-Gordon equation as characteristics.

(NLKG) $$\Box u + cu + bu^2 + au^3 = g \quad \text{on } (x, t) \in \Omega \times (0, T_0),$$
$$u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0 \quad \text{and} \quad u_t|_{t=0} = v_0.$$

The meaning of the characteristic, the definition of functional derivatives and the terminology used here, are explained precisely in Inoue [3].

Let $\{w_j\}$ be a complete orthonormal basis in L^2, dense in $\mathring{H}^1 \cap H^2$ such that (1) $w_j(x) \in L^2_r \cap \mathring{H}^1$, $\partial_x^\alpha w_j(x) \in L^2_r$ for $|\alpha| \le 2$ and (2) $(1+|x|^2)^{r/2} w_j(x) \in L^\infty$, $(1+|x|^2)^{r/2} \partial w_j(x)/\partial x^k \in L^\infty$ for some $r > 0$. We put $\pi_m u = \sum_{j=1}^m \langle u, w_j \rangle w_j$.

Let $u_m(t) \in C^2([0, T_0); \pi_m V)$ be the Galerkin approximation of NLKG which satisfies

$$\frac{d}{dt} U_m(t) = \Pi_m L(U_m(t)) + \Pi_m G(t) \quad \text{with} \quad U_m(0) = \Pi_m U_0, \quad U_0 = {}^t(u_0, v_0)$$

where $\Pi_m U = {}^t(\pi_m u, \pi_m v)$, $U_m(t) = {}^t(u_m(t), v_m(t))$, $L(U) = {}^t(v, \Delta u - f(u))$, $G(t) = {}^t(0, g(t))$.

Lemma 1. *Assume* (AS 0). *For any* $\varepsilon > 0$, $t > 0$, *we have*

$$H(u_m(t), v_m(t)) \le e^{\varepsilon t}\left[H(u_{0m}, v_{0m}) + \frac{1}{2\varepsilon}\int_0^t |g(s)|^2 ds\right].$$

Moreover, putting $C_{t,\varepsilon} = 1 + (2t^2 + \varepsilon t)e^{t^2 + \varepsilon t}$, *we get*

$$E_\kappa(U_m(t)) \equiv \frac{1}{2}|\dot{u}_m(t)|_2^2 + \frac{1}{2}\|u_m(t)\|_1^2 + \kappa |u_m(t)|_4^4 \le C_{t,\varepsilon}\left[E_*(U_m(0)) + \frac{1}{2\varepsilon}\int_0^t |g(s)|^2 ds\right].$$

Put $\Pi_m X = {}^t(\pi_m V \times \pi_m L^2)$, $X_\infty = \cup_{m=1}^\infty \Pi_m X$, $\Pi_m \tilde{X} = {}^t(\pi_m \tilde{V} \times \pi_m H^{-\delta})$, $\tilde{X}_\infty = \cup_{m=1}^\infty \Pi_m \tilde{X}$, and $\tilde{X}_\infty^* = \cup_{m=1}^\infty \Pi_m \tilde{X}^*$. We define an operator from $\Pi_m X$ to $C([0, T_0); \Pi_m X)$ by $S_m(t)(\Pi_m U_0) = {}^t(u_m(t), \dot{u}_m(t))$ for $U_0 \in X$. For any measure μ_0 on X and $\omega \in \mathcal{B}(X)$, we define, $\mu_0^{(m)}(\omega) \equiv \mu_0(\Pi_m^{-1}(\omega \cap \Pi_m X))$, $\mu^{(m)}(t, \omega) \equiv \mu_0^{(m)}(S_m(t)^{-1}\omega)$. Clearly, $\mu_0^{(m)}(dU)$ and $\mu^{(m)}(t, dU)$ are concentrated on $\Pi_m X = \Pi_m \tilde{X}$.

Lemma 2. *For any test functional* Φ *with compact support in* t, *we have*

$$\int_0^{T_0}\int_X \frac{\partial \Phi(t, U)}{\partial t}\mu^{(m)}(t, dU)dt + \int_X \Phi(0, U)\mu_0^{(m)}(dU)$$
$$= -\int_0^{T_0}\int_X [\langle \Delta u - f(u) + g(t), \Phi_v(t, U)\rangle + \langle v, \Phi_u(t, U)\rangle]\mu^{(m)}(t, dU)dt.$$

Defining the Fourier-Stieltjes transform of $\mu^{(m)}(t, dU)$ and the operator $L(\delta/\delta\Xi)$ by

$$W^{(m)}(t, \Xi) = \int_X e^{i\langle \Xi, U\rangle}\mu^{(m)}(t, dU) = \int_X e^{i\langle \Pi_m \Xi, U\rangle}\mu^{(m)}(t, dU)$$

and

$$L\left(\frac{\delta}{\delta \Xi}\right)W^{(m)}(s, \Xi) = \int_X e^{i\langle \Pi_m \Xi, U\rangle}\langle \Pi_m \Xi, L(U)\rangle \mu^{(m)}(s, dU),$$

we have

Lemma 3. *Under Assumption* (AS 0), *we have*

$$\dot{W}^{(m)}(t, \Xi)=iL\left(\frac{\delta}{\delta\Xi}\right)W^{(m)}(t, \Xi)+i\langle\Xi, G(t)\rangle W^{(m)}(t, \Xi)$$
$$\text{for } \Xi \in \Pi_k \tilde{X}^*,\ k\leq m$$

Moreover, we remark

Lemma 4. (1) *X is compactly imbedded in $\tilde{X}$.*

(2) *There exists a constant C such that*

$$1+\Lambda(U)\leq C(1+E_\kappa(U))^\beta \qquad \text{where } \begin{cases}\beta=3/4 & \text{for } \kappa>0,\\ \beta=3/2 & \text{for } \kappa=0,\ d\leq 3.\end{cases}$$

Proceeding as in Vishik and Komec [4], we get

Lemma 5. *$W^{(m)}(t, \Xi)$ forms a equicontinuous and equibounded set on $C([0, T_0)\times Y^*)$ where $Y^*=L^2\times V$.*

From this, there exist $W(t, \Xi)$ and a subsequence $W^{(m')}(t, \Xi)$ such that $W^{(m')}(t, \Xi)$ converges uniformly to $W(t, \Xi)$. Using the Prokhorov theorem and modifying a little the arguments in [4], we have

Proposition. (1) *For any t, there exists a measure $\mu(t, dU)$ such that $(1+\Lambda(U))\mu^{(m')}(t, dU)$ converges weakly to $(1+\Lambda(U))\mu(t, dU)$ on $\tilde{X}$. And this implies that $\mu^{(m')}(t, dU)$ itself converges weakly to $\mu(t, dU)$ on $\tilde{X}$.*

(2) *Any weak limit $\mu(t, dU)$ of measures $\mu^{(m')}(t, dU)$ has the Fourier-Stietjes transform $\hat{\mu}(t, \Xi)=W(t, \Xi)$ for $\Xi \in Y^*$, $t \in [0, T_0)$.*

(3) *For any $t \in [0, T_0)$, $\mu(t, \tilde{X}\backslash X)=0$.*

Lemma 6. *For $\Xi \in \tilde{X}^*_\infty$, $\int_X e^{i\langle\Xi, U\rangle}\langle\Xi, L(U)\rangle\mu^{(m')}(t, dU)$, the sequence of continuous functions of $t \in [0, T_0]$ is uniformly bounded, and for any t, it converges to $\int_X e^{i\langle\Xi, U\rangle}\langle\Xi, L(U)\rangle\mu(t, dU)$ as $m'\to\infty$.*

Combining these with the arguments in Foiaş [1], we get Theorem A. On the other hand, by the conditions for $W_0(\Xi)$, we may suppose that there exists a measure $\mu_0(dU)$ on X satisfying $\hat{\mu}_0(\Xi)=W_0(\Xi)$ and (AS 1). Remarking the facts explained in Foiaş [2] and Inoue [3], we may prove Theorem B.

Remark. Detailed proofs with other topics will be published elsewhere in the near future.

References

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