94. *On the Automorphism Groups of Edge-coloured Digraphs*

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1. Introduction. For any finite group $G=\{g_1, g_2, \cdots, g_q\}$, we construct an edge-coloured strongly connected digraph $\Delta=\Delta(G)$ with the vertex-set $V\Delta=\{g_1, g_2, \cdots, g_q\}$ such that for any two vertices u and v of Δ both (u, v) and (v, u) are directed edges of Δ and they are coloured with colours $u^{-1}v$ and $v^{-1}u$ respectively. Then the (colour-preserving) automorphism group $\mathrm{Aut}\,\Delta$ of Δ on $V\Delta$ is isomorphic to the regular representation [4] of G as a permutation group ([5, p. 96, Lemma 3.1]). On the other hand, Frucht [1] and Sabidussi [2] proved the following: For any finite group G and any integer $k\geqq 3$ there exist infinitely many connected k-regular (undirected) graphs Γ in which $V\Gamma$ has a disjoint union decomposition $V\Gamma=\sum_{i=1}^{q} V_i$ ($q=|G|$) such that the automorphism group $\mathrm{Aut}\,\Gamma$ of Γ acts faithfully on the set $\{V_1, V_2, \cdots, V_q\}$ by the natural action and the permutation group derived by its action is isomorphic to the regular representation of G as a permutation group.

In this peper we shall extend the above. Let Δ be an edge-coloured digraph and C be the set of colours c with which at least one directed edge of Δ is coloured. We define a uniquely definite positive integer $\lambda(\Delta)$ as follows. For any vertex x of Δ and c in C let $\lambda_{\mathrm{in}}(x; c)$ denote the number of directed edges with colour c having x as head and $\lambda_{\mathrm{out}}(x; c)$ denote the number of directed edges with colour c having x as tail. We define $\lambda_{\max}(\Delta)=\max\{\lambda_{\mathrm{in}}(x; c), \lambda_{\mathrm{out}}(x; c) : x\in V\Delta, c\in C\}$ and $\lambda(\Delta)=\max\{\lambda_{\max}(\Delta)+1, 3\}$. The purpose of this paper is to prove

Theorem. *Let Δ be an edge-coloured weakly connected digraph with $|V\Delta|=n$. Then for any integer $k\geqq\lambda(\Delta)$ there exist infinitely many connected k-regular (undirected) graphs Γ in which $V\Gamma$ has a disjoint union $V\Gamma=\sum_{i=1}^{n} V_i$ such that $\mathrm{Aut}\,\Gamma$ acts faithfully on the set $\{V_1, V_2, \cdots, V_n\}$ by the natural action and the permutation group derived by its action is isomorphic to the (colour-preserving) automorphism group $\mathrm{Aut}\,\Delta$ of Δ on $V\Delta$ as a permutation group.*

2. Preliminaries. Unless stated otherwise, all graphs are finite, undirected, simple and loopless. If an edge e joins two vertices u and v, we write $e=[u, v]=[v, u]$. If $\mathrm{Aut}\,\Gamma=1$, Γ is called asymmetric.

Now we introduce a notion of the type [1] $(a_1, a_2, \cdots, a_r)$ ($r=m(m-1)/2$) of a vertex v of valency m in a graph Γ. Let $u_1, u_2, \cdots, u_m$ be the adjacent vertices of v. We define the number α_{ij} ($i<j$) as follows:

$\alpha_{ij}=$the minimum length of circuits which contain the two edges $[u_i, v]$

and $[v, u_j]$ if there exists such a circuit,
$=\infty$ otherwise.

By ranging $m(m-1)/2$ numbers of α_{ij}'s in increasing order, we get the type $(a_1, a_2, \cdots, a_r)$ of v, where $r=m(m-1)/2$, $a_1 \leqq a_2 \leqq \cdots \leqq a_r$ and $\{a_1, a_2, \cdots, a_r\} = \{\alpha_{ij}: 1 \leqq i < j \leqq m\}$.

We shall make substantial use of methods of Sabidussi [2, 3]: For graphs $\Gamma_1, \Gamma_2, \cdots$ and Γ_h we define the product $\prod_{i=1}^h \Gamma_i$ by

$V(\prod_{i=1}^h \Gamma_i) = \prod_{i=1}^h V\Gamma_i$ (the cartesian product of the sets $V\Gamma_i$),

$E(\prod_{i=1}^h \Gamma_i) = \{[(u_1, u_2, \cdots u_h), (v_1, v_2, \cdots, v_h)]: \{i: u_i \neq v_i, 1 \leqq i \leqq h\}$ is a one-element set $\{j\}$ satisfying $[u_j, v_j] \in E\Gamma_j\}$.

A graph Γ is called prime if Γ is non-trivial and if $\Gamma \cong \Lambda \times \Pi$ implies that Λ or Π is trivial, where a trivial graph is a vertex-graph. Two graphs Γ_1 and Γ_2 are called relatively prime if $\Gamma_1 \cong \Gamma_1' \times \Pi$ and $\Gamma_2 \cong \Gamma_2' \times \Pi$ imply that Π is a trivial graph. We say that a connected graph Γ can be decomposed into prime factors if there exist connected prime graphs $\Gamma_1, \Gamma_2, \cdots, \Gamma_r$ satisfying $\Gamma \cong \prod_{i=1}^r \Gamma_i$.

We suppose that any digraph Δ has no loops. We denote a directed edge whose tail is u and whose head is v by (u, v). An edge-coloured digraph Δ is a digraph Δ together with a function $\phi: E\Delta \to C$ which maps $E\Delta$ into a set C of colours. Of course any automorphism of an edge-coloured digraph Δ must preserve colours.

Lemma 1. *Let u, v be vertices of graphs Γ_1, Γ_2 respectively. Then the valency of (u, v) in $\Gamma_1 \times \Gamma_2$ is the sum of the valencies of u and v.*

Lemma 2 [2]. *If in a connected graph Γ there is an edge which is not contained in a 4-cycle, then Γ is prime.*

Proposition 1 [3]. *Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_h$ be connected relatively prime graphs. Then*

$$\mathrm{Aut}\,(\prod_{i=1}^h \Gamma_i) \cong \prod_{i=1}^h \mathrm{Aut}\,\Gamma_i.$$

Proposition 2 [3]. *If a connected graph Γ has a prime factor decomposition, then the prime factor decomposition of Γ is unique up to isomorphisms.*

Corollary 1. *Any connected graph has the unique prime factor decomposition up to isomorphisms.*

The proofs of the following Lemmas 3, 4, 5 and 6 are easy.

Lemma 3. *There is a connected 3-regular asymmetric graph Γ of girth 5.*

Lemma 4. *For any even integer $n \geqq 26$ there is a connected 3-regular asymmetric prime graph Γ of girth 4 with $|V\Gamma| = n$ such that $\Gamma - e$ is a connected graph of girth 4 for some edge e of Γ.*

Lemma 5. *There exist infinitely many connected 4-regular asymmetric prime graphs Γ of girth 4 such that $\Gamma - e$ is a connected graph of girth 4 for some edge e of Γ.*

Lemma 6. *There exist infinitely many connected 5-regular asymmetric prime graphs Γ of girth 4 such that $\Gamma - e$ is a connected graph of girth 4* *for some edge e of Γ.*

Proposition 3. *For any integer $m\geqq 3$ there exist infinitely many connected m-regular asymmetric graphs Γ of girth 4 such that $\Gamma-e$ is a connected graph of girth 4 for some edge e of Γ.*

Proof. Let t be an integer with $t\equiv m \pmod 3$ and $3\leqq t\leqq 5$. Let us set $q=(m-t)/3$. By Lemmas 4, 5 and 6 there exist non-isomorphic $q+1$ connected asymmetric prime graphs $\Gamma_0, \Gamma_1, \cdots, \Gamma_q$ of girth 4 such that Γ_0 is t-regular and Γ_i is 3-regular $(1\leqq i\leqq q)$ and that Γ_0-e_0 is a connected graph of girth 4 for some edge e_0 of Γ_0. Then the product $\Gamma=\Gamma_0\times\Gamma_1\times\cdots\times\Gamma_q$ is a connected m-regular graph of girth 4 such that $\Gamma-e$ is a connected graph of girth 4 for some edge e of Γ. By Proposition 1 Γ is asymmetric.

Corollary 2. *For any integer $m\geqq 3$ there exist infinitely many connected asymmetric graphs Γ of girth 4 such that just two vertices of Γ have valency $m-1$ and every other vertex of Γ has valency m.*

Proof. Let Γ be a connected m-regular asymmetric graph of girth 4 such that $\Gamma-e$ is a connected graph of girth 4 for some edge e of Γ. Since $\mathrm{Aut}(\Gamma-e)$ is a subgroup of $\mathrm{Aut}\,\Gamma$, $\mathrm{Aut}(\Gamma-e)=1$ holds. Hence by Proposition 3 we complete the proof.

3. Proof of Theorem. Let Δ be an edge-coloured weakly connected digraph with $|V\Delta|=n$ and k be any integer with $k\geqq\lambda(\Delta)$. Let $E_1, E_2, \cdots, E_s$ be the all orbits [4] of $\mathrm{Aut}\,\Delta$ on $E\Delta$. Obviously two directed edges which are in the same orbit E_i have the same colour, but two directed edges which are in different orbits E_i and E_j do not necessarily have different colours. Now let $C=\{c_1, c_2, \cdots, c_s\}$ be a set of (different) s colours. We paint the directed edges in E_i the colour c_i for convenience $(i=1,2,\cdots,s)$. We remark that the permutation group $\mathrm{Aut}\,\Delta$ on $V\Delta$ is unchangeable and $\lambda(\Delta)$ does not become larger by the painting. Hence from now on we prove the theorem on Δ which has been changed as above.

Let b be a positive integer satisfying $4s<k(k-1)^b$. For any $x\in V\Delta$ we first define a graph Ω_x by

$$V\Omega_x=\{x_i:\ 1\leqq i\leqq k\}\,\dot{\cup}\,\{x_{ij}:\ 0\leqq i\leqq b,\ 1\leqq j\leqq k(k-1)^i\}$$
$$\dot{\cup}\,\{x_{b+1\,j}:\ 1\leqq j\leqq k(k-1)^b\}\ \text{(disjoint union)},$$
$$E\Omega_x=\{[x_i, x_j]:\ 1\leqq i<j\leqq k\}\cup\{[x_i, x_{0i}]:\ 1\leqq i\leqq k\}\cup\{[x_{ij}, x_{i+1\,h}]:$$
$$0\leqq i\leqq b-1,\ 1\leqq j\leqq k(k-1)^i,\ (j-1)(k-1)+1\leqq h\leqq j(k-1)\}$$
$$\cup\{[x_{bj}, x_{b+1\,j}]:\ 1\leqq j\leqq k(k-1)^b\}.$$

Then Ω_x is a connected graph in which the vertices x_{bj} $(j=1,2,\cdots,k(k-1)^b)$ have valency 2, the vertices $x_{b+1\,j}(j=1,2,\cdots,k(k-1)^b)$ have valency 1 and the other vertices have valency k. Let us set

$$p=k(k-1)^b/2,$$
$$q=\{k(k-1)^b(k-2)+k(k-1)^b(k-1)-2\sum_{i=1}^{s}\lambda_{\mathrm{out}}(x;c_i)-2\sum_{i=1}^{s}\lambda_{\mathrm{in}}(x;c_i)\}/2.$$

By Corollary 2 we have non-isomorphic q graphs $\Lambda(x_{bj}, r)$ $(j=1,2,\cdots,p;\ r=1,2,\cdots,k-2)$, $\Lambda(x_{b+1\,j}, r)$ $(j=1,2,\cdots,s;\ r=1,2,\cdots,(k-1)-\lambda_{\mathrm{out}}(x;c_j))$, $\Lambda(x_{b+1\,j}, r)$ $(j=s+1,s+2,\cdots,2s;\ r=1,2,\cdots,(k-1)-\lambda_{\mathrm{in}}(x;c_{j-s}))$ and

$\Lambda(x_{b+1\,j}, r)$ $(j=2s+1, 2s+2, \cdots, p;\ r=1, 2, \cdots, k-1)$
each of which is a connected asymmetric graph Λ of girth 4 such that just two vertices of Λ have valency $k-1$ and every other vertex of Λ has valency k. Let $u(x_{hj}, r)$ and $u'(x_{hj}, r)$ $(b \leqq h \leqq b+1)$ be the vertices of valency $k-1$ of $\Lambda(x_{hj}, r)$. Next we define a graph Π_x by

$$V\Pi_x = V\Omega_x \dot{\cup} (\sum_{j,r} V(\Lambda(x_{bj}, r))) \dot{\cup} (\sum_{j,r} V(\Lambda(x_{b+1\,j}, r))) \text{ (disjoint union)},$$

$$E\Pi_x = E\Omega_x \cup (\sum_{j,r} E(\Lambda(x_{bj}, r))) \cup (\sum_{j,r} E(\Lambda(x_{b+1\,j}, r))) \cup \{[x_{bj}, u(x_{bj}, r)], [x_{b\,j+p}, u'(x_{bj}, r)] : 1 \leqq j \leqq p, 1 \leqq r \leqq k-2\} \cup \{[x_{b+1\,j}, u(x_{b+1\,j}, r)], [x_{b+1\,j+p}, u'(x_{b+1\,j}, r)] : 1 \leqq j \leqq s, 1 \leqq r \leqq (k-1) - \lambda_{\text{out}}(x; c_j)\} \cup \{[x_{b+1\,j}, u(x_{b+1\,j}, r)], [x_{b+1\,j+p}, u'(x_{b+1\,j}, r)] : s+1 \leqq j \leqq 2s, 1 \leqq r \leqq (k-1) - \lambda_{\text{in}}(x; c_{j-s})\} \cup \{[x_{b+1\,j}, u(x_{b+1\,j}, r)], [x_{b+1\,j+p}, u'(x_{b+1\,j}, r)] : 2s+1 \leqq j \leqq p, 1 \leqq r \leqq k-1\}.$$

Then Π_x is a connected graph in which there exists the unique complete subgraph with k vertices induced by $\{x_1, x_2, \cdots, x_k\}$ and both vertices $x_{b+1\,j}$ and $x_{b+1\,j+p}$ have valency $k-\lambda_{\text{out}}(x; c_j)$ for $j=1, 2, \cdots, s$, both vertices $x_{b+1\,j}$ and $x_{b+1\,j+p}$ have valency $k-\lambda_{\text{in}}(x; c_{j-s})$ for $j=s+1, s+2, \cdots, 2s$ and every other vertex has valency k. By Corollary 2 we may assume that every Π_x $(x \in V\Delta)$ is asymmetric and for $x, y \in V\Delta$, Π_x is isomorphic to Π_y if and only if there is an automorphism σ of Δ with $\sigma(x)=y$. Moreover we may assume that if Π_x is isomorphic to Π_y for $x, y \in V\Delta$, then the isomorphism from Π_x to Π_y has the correspondence of $x_{b+1\,j}$ to $y_{b+1\,j}$ $(1 \leqq j \leqq k(k-1)^b)$. Last we define a graph Γ by

$$V\Gamma = \sum_{x \in V\Delta} V\Pi_x \text{ (disjoint union)},$$

$$E\Gamma = (\sum_{x \in V\Delta} E\Pi_x) \cup \{[x_{b+1\,i}, y_{b+1\,s+i}], [x_{b+1\,i+p}, y_{b+1\,s+i+p}] : (x, y) \in E\Delta, \text{ the colour of } (x, y) \text{ is } c_i\}.$$

Then Γ is a connected k-regular graph such that $\mathrm{Aut}\,\Gamma$ acts faithfully on the set $\{V\Pi_x : x \in V\Delta\}$ by the natural action and the permutation group derived by its action is isomorphic to the automorphism group $\mathrm{Aut}\,\Delta$ of Δ on $V\Delta$ as a permutation group by the correspondence of $V\Pi_x$ to x $(x \in V\Delta)$.

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