90. *Differential Inequalities and Carathéodory Functions*

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Let P be the class of functions $p(z)$ which are analytic in the unit disk $E=\{z: |z|<1\}$, with $p(0)=1$ and $\operatorname{Re} p(z)>0$ in E.

If $p(z)\in P$, we say $p(z)$ a Carathéodory function. It is well known that if $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is analytic in E and $\operatorname{Re} f'(z)>0$ in E, then $f(z)$ is univalent in E [2, 7].

Ozaki [6, Theorem 2] extended the above result to the following:

If $f(z)$ is analytic in a convex domain D and

$$\operatorname{Re}(e^{i\alpha} f^{(p)}(z))>0 \qquad \text{in } D$$

where α is a real constant, then $f(z)$ is at most p-valent in D.

This shows that if $f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n$ is analytic in E and

$$\operatorname{Re} f^{(p)}(z)>0 \qquad \text{in } E,$$

then $f(z)$ is p-valent in E.

Nunokawa [3] improved the above result to the following:

Let $p\geqq 2$. If $f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n$ is analytic in E and

$$|\arg f^{(p)}(z)|<\frac{3}{4}\pi \qquad \text{in } E,$$

then $f(z)$ is p-valent in E.

Definition. Let $F(z)$ be analytic and univalent in E and suppose that $F(E)=R$. If $f(z)$ is analytic in E, $f(0)=F(0)$, and $f(E)\subset R$, then we say that $f(z)$ is subordinate to $F(z)$ in E, and we write

$$f(z)\prec F(z).$$

In this paper, we need the following lemmata.

Lemma 1. *If $p(z)$ is analytic in E, with $p(0)=1$ and*

$$\operatorname{Re}(p(z)+zp'(z))>\beta \qquad \textit{in } E,$$

where $\beta<1$, then we have

(1) $$\operatorname{Re} p(z)>(1-\beta)\log\frac{4}{e}+\beta \qquad \textit{in } E.$$

Proof. Let us put

$$g(z)=\frac{1}{1-\beta}(p(z)+zp'(z)-\beta)$$
$$=\frac{1}{1-\beta}((zp(z))'-\beta).$$

Then we have

$$g(z)\in P.$$

This shows that

$$g(z)=\frac{1}{1-\beta}((zp(z))'-\beta)\prec\frac{1+z}{1-z}$$

and it follows that

(2) $$(zp(z))'\prec(1-\beta)\frac{1+z}{1-z}+\beta.$$

Then we have

$$zp(z)=\int_0^z (tp(t))'dt,$$

and therefore, we have

$$\begin{aligned}p(z)&=\frac{1}{z}\int_0^z (tp(t))'dt\\&=\frac{1}{re^{i\theta}}\int_0^r (tp(t))'e^{i\theta}d\rho\\&=\frac{1}{r}\int_0^r (tp(t))'d\rho\end{aligned}$$

where $z=re^{i\theta}$, $0<r<1$, $t=\rho e^{i\theta}$ and $0\leqq\rho\leqq r$.

From [1, Theorem 7, p. 84], (2) and applying the same method as in the proof of [4, Main theorem], we have

$$\begin{aligned}\operatorname{Re} p(z)&=\frac{1}{r}\int_0^r \operatorname{Re}(tp(t))'d\rho\\&\geqq\frac{1}{r}\int_0^r\left[(1-\beta)\frac{1-\rho}{1+\rho}+\beta\right]d\rho\\&=\frac{1}{r}[(1-\beta)(-r+2\log(1+r))+\beta r]\\&=(1-\beta)\,[2\log(1+r)^{1/r}-1]+\beta\\&\geqq(1-\beta)[2\log 2-1]+\beta\\&=(1-\beta)\log\frac{4}{e}+\beta\end{aligned}$$

for $0<|z|=r<1$. This completes our proof.

From Lemma 1, we easily have the following result:

Lemma 2. *If $p(z)$ is analytic in E, with $p(0)=1$ and*

$$\operatorname{Re}(p(z)+zp'(z))>-\frac{\log(4/e)}{2\log(e/2)}\qquad in\ E,$$

then $p(z)\in P$ or $p(z)$ is a Carathéodory function.

Proof. Putting the right-hand side of (1) be zero, then we have the equation

$$(1-\beta)\log\frac{4}{e}+\beta=0$$

and that

$$\beta=-\frac{\log(4/e)}{2\log(e/2)}.$$

This shows that

$$\operatorname{Re}(p(z)+zp'(z))>-\frac{\log(4/e)}{2\log(e/2)} \quad \text{in } E$$

implies $\operatorname{Re} p(z)>0$ in E.

Lemma 3. *Let $f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$ be analytic in E and if there exists a $(p-k+1)$-valent starlike function $g(z)=z^{p-k+1}+\sum_{n=p-k+2}^{\infty}b_nz^n$ that satisfies*

$$\operatorname{Re}\frac{zf^{(k)}(z)}{g(z)}>0 \quad \textit{in } E,$$

then $f(z)$ is p-valent in E.

We own this lemma to [5, Theorem 8].

Main theorem. Let $p\geqq 2$. If $f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$ is analytic in E and

$$\operatorname{Re} f^{(p)}(z)>-\frac{\log(4/e)}{2\log(e/2)}p\,! \quad \text{in } E, \tag{3}$$

then $f(z)$ is p-valent in E.

Proof. Let us put

$$p(z)=f^{(p-1)}(z)/(p\,!\,z).$$

Then, from the assumption (3) and by an easy calculation, we have

$$\begin{aligned}\operatorname{Re}(p(z)+zp'(z))&=\operatorname{Re}(f^{(p)}(z)/p\,!)\\ &>-\frac{\log(4/e)}{2\log(e/2)} \quad \text{in } E,\end{aligned}$$

and $p(0)=1$. Then, from Lemma 2, we have

$$\operatorname{Re} p(z)=\frac{1}{p\,!}\operatorname{Re}\frac{f^{(p-1)}(z)}{z}>0 \quad \text{in } E.$$

This shows that

$$\operatorname{Re}\frac{zf^{(p-1)}(z)}{z^2}>0 \quad \text{in } E. \tag{4}$$

It is trivial that $g(z)=z^2$ is 2-valently starlike in E. Therefore, from Lemma 3 and (4), we have that $f(z)$ is p-valent in E.

This completes our proof.

References

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