89. Some Remarks on Index and Entropy for von Neumann Subalgebras

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(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 12, 1989)

In the present note, we introduce two notions, i.e. *finite type* of inclusion relation of von Neumann algebras and *indicial derivative*. The former is a generalization of index finite type and entropy finite type. The latter is a substitute of the index initiated by V. Jones [3] and extended by H. Kosaki [6]. The aim of the present note is to report that the indicial derivative produces both of the index and Pimsner-Popa's entropy [7].

1. Let $M \supset N$ be a pair of von Neumann algebras on a Hilbert space H. The representation space H is assumed to be separable throughout the present note. For the pair $M \supset N$, let $P(M, N)$ denote the set of all faithful normal semifinite N-valued weights on M. Moreover, set $P_1(M, N) = \{E \in P(M, N): \sigma_t^E = id\}$ and $E_1(M, N) = \{E \in P_1(M, N): E(1) = 1\}$. $P(M, C)$ [resp. $E_1(M, C)$] is often denoted by $P(M)$ [resp. $E_1(M)$]. For each $E \in P(M, N)$, let E^c denote the restriction of E to $N' \cap M$ and let E^{-1} denote the Haagerup correspondent of E, uniquely determined by the equation of spatial derivative $\Delta((\varphi \circ E)/\psi) = \Delta(\varphi/(\psi \circ E^{-1}))$ for $\varphi \in P(N)$ and $\psi \in P(M')$. For more details, refer to [1], [2].

Lemma 1. *Let $M \supset N$ be as above. Then, there exists $E \in E_1(M, N)$ with $(E^{-1})^c \in P_1(N' \cap M, Z(M))$ if and only if $E_1(M, N) \neq \emptyset$ and $E_1(N', M') \neq \emptyset$.*

When a pair $M \supset N$ of von Neumann algebras satisfies the conditions in Lemma 1, we say that the inclusion relation $R(M, N)$ is *of finite type*. Let $ET(M, N)$ denote the set of all pairs (E, τ) where $E \in E_1(M, N)$ and $\tau \in E_1(N' \cap M)$ such that $\tau \circ E^c = \tau$. Then, if $R(M, N)$ is of finite type, $ET(M, N) \neq \emptyset$, and for each $(E, \tau) \in ET(M, N)$, one can take $E' \in E_1(N', M')$, uniquely determined by the condition that $\tau \circ (E')^c = \tau$ and we call it *standard correspondent* of E w.r.t. τ. In this case, a generalized Pedersen-Takesaki's derivative dE^{-1}/dE' is well defined by $dE^{-1}/dE' = d(\varphi \circ E^{-1})/d(\varphi \circ E')$ for $\varphi \in P(M')$ because the derivative $d(\varphi \circ E^{-1})/d(\varphi \circ E')$ does not depend on the choice of $\varphi \in P(M')$. Since this derivative dE^{-1}/dE' is determined for $(E, \tau) \in ET(M, N)$, we denote it by $I_\tau^E(M|N)$ and we call it *indicial derivative* of E w.r.t. τ.

Lemma 2. *Let $M \supset N$ be a pair of von Neumann algebras such that $R(M, N)$ is of finite type. Then, for $(E, \tau) \in ET(M, N)$, the indicial derivative $I_\tau^E(M|N)$ is a positive selfadjoint operator affiliated with the center $Z(N' \cap M)$ of $N' \cap M$ such that $I_\tau^E(M|N) = d(\tau \circ (E^{-1})^c)/d\tau \geq 1$.*

2. For a pair $M \supset N$ of von Neumann algebras and $E \in E_1(M, N)$,

index $E=E^{-1}(1)$ is defined as an extended positive element of $Z(M)$, according to Kosaki's definition [6]. We note that, if $Z(N'\cap M)=C$, $E_1(M,N)$ consists of a unique element E_0 and index $E_0=I_\tau^{E_0}(M|N)$ $(\tau=E_0^c)$, which we denote by $[M:N]$ according to Jones' notation. For a pair of finite von Neumann algebras with $\tau\in E_1(M)$, the relative entropy $H_\tau(M|N)$ has been defined by Pimsner-Popa [7], associated with $E\in E_1(M,N)$ such that $\tau\circ E=\tau$. They have obtained a remarkable result: $H_\tau(M|N)=\log[M:N]$ for a pair $M\supset N$ of type II_1 factors with $Z(N'\cap M)=C$. The next theorem is a generalization of this result.

Theorem 3. (i) *Let $M\supset N$ be a pair of von Neumann algebras. If there exists $E\in E_1(M,N)$ such that* index $E\in Z(M)^+$, *then, $R(M,N)$ is of finite type. If $R(M,N)$ is of finite type, then, for $(E,\tau)\in ET(M,N)$,* index $E=E'(I_\tau^E(M|N))$ *where E' is the standard correspondent of E w.r.t. τ.*

(ii) *Let $M\supset N$ be a pair of von Neumann algebras of type II_1. If $H_\tau(M|N)<+\infty$ for some $\tau\in E_1(M)$, $R(M,N)$ is of finite type. If $R(M,N)$ is of finite type, then, for $(E,\tau)\in ET(M,N)$, $H_\tau(M|N)=\tau(\log I_\tau^E(M|N))$.*

We note that it often occurs that index $E=\infty\cdot 1$ or $H_\tau(M|N)=+\infty$ even if $R(M,N)$ is of finite type. Theorem 3 is of use in these cases too and we observe from this theorem that the indicial derivative $I_\tau^E(M|N)$ works as a kind of index of $E\in E_1(M,N)$ which contains more informations than the original index or entropy. Moreover, several formulas on index and entropy immediately follow from those of indicial derivative. As a remarkable application, some answers can be afforded to the following problem: When does the equality $H_\tau(M|N)=H_\tau(M|L)+H_\tau(L|N)$ hold true for an algebra L such that $M\supset L\supset N$?

Corollary 4. *Let $M\supset N$ be a pair of von Neumann algebras of type II_1 with $\tau\in E_1(M)$.*

(i) *For such a von Neumann algebra L that $M\supset L\supset N$, the followings are equivalent in the case that $H_\tau(M|N)<+\infty$.*

(a) $H_\tau(M|N)=H_\tau(M|L)+H_\tau(L|N)$.

(b) *$E'=E_1'\circ E_2'$ for $E\in E_1(M,N)$, $E_1\in E_1(M,L)$ and $E_2\in E_1(L,N)$, all of which are determined by keeping the trace τ fixed.*

(c) *$\sigma_t^{\varphi\circ E'}(L')=L'$ for some $\varphi\in P(M')$ and $E\in E_1(M,N)$ such that $\tau\circ E=\tau$.*

(ii) *If the algebra L is given by $M\cap A'$ or $N\vee A$ for a subalgebra A in $N'\cap M$, $H_\tau(M|N)=H_\tau(M|L)+H_\tau(L|N)$ holds true.*

The statement (ii) of Theorem 3 leads us to define an abstract entropy $K_\tau^E(M|N)=\tau(\log d(\tau\circ E^{-1})/d\tau)$ in a sense of extended operator calculus. The proof of the statement (ii) has been done by checking the equality $H_\tau(M|N)=K_\tau^E(M|N)$ in each step of the reduction ([4], [5]). Its intrinsic proof will be expected.

3. We shall describe some fundamental properties on finiteness of $R(M,N)$ and so on.

Proposition 5. *Let $M\supset N$ be a pair of von Neumann algebras.*

(i) *The finiteness of $R(M,N)$ does not depend on the representation space.*

(ii) *$R(M,N)$ is of finite type if and only if so is $R(N',M')$. If this is the case, for $(E,\tau)\in ET(M,N)$, $I_\tau^E(M|N)=I_\tau^{E'}(N'|M')$ and $K_\tau^E(M|N)=K_\tau^{E'}(N'|M')$.*

(iii) *When either M or N is a factor, $R(M,N)$ is of finite type if and only if $N'\cap M$ is atomic and $[M_p:N_p]<+\infty$ for all atoms $p\in Z(N'\cap M)$.*

(iv) *If $R(M,N)$ is of finite type, $N'\cap M$ must be a type I algebra of finite type.*

Proposition 6. (i) *Let $M\supset N$ be a pair of von Neumann algebras such that $N'\cap M$ is atomic and let $\{e_i\}_{i\in I}$ [resp. $\{f_j\}_{j\in J}$] denote the set of all atoms of $Z(M)$ [resp. $Z(N)$]. Then, we obtain, for $(E,\tau)\in ET(M,N)$,*

$$I_\tau^E(M|N)=\sum_{i,j}(\tau(e_i)\tau(f_j)/\tau(e_if_j)^2)I_{\tau_{ij}}^{E_{ij}}(M_{e_if_j}|N_{e_if_j})e_if_j,$$

where (i,j) runs over $e_if_j\neq 0$, τ_{ij} is the reduced normalized trace of τ, and $E_{ij}\in E_1(M_{e_if_j}|N_{e_if_j})$ is given by $\tau_{ij}\circ E_{ij}^c=\tau_{ij}$.

(ii) *Let $M\supset N$ be a pair of factor-subfactor such that $N'\cap M$ is atomic and let $\{p_k\}_{k\in K}$ denote the set of all atoms of $Z(N'\cap M)$. Then, we obtain, for $(E,\tau)\in ET(M,N)$,*

$$I_\tau^E(M|N)=\sum_{k\in K}([M_{p_k}:N_{p_k}]/\tau(p_k)^2)p_k.$$

Corollary 7. *Under the same situations as the above* (i), (ii) *respectively, we get the following formulas.*

(i) $$\operatorname{index}E=\sum_{i,j}((\tau(f_j)/\tau(e_if_j))\operatorname{index}E_{ij})e_i$$

$$K_\tau^E(M|N)=\sum_{i,j}\{\tau(e_if_j)K_{\tau_{ij}}^{E_{ij}}(M_{e_if_j}|N_{e_if_j})+2\eta(\tau(e_if_j))\}-\sum_i\eta(\tau(e_i))-\sum_j\eta(\tau(f_j)),$$

where $\eta(t)=-t\log t$ for $t>0$.

(ii) $$\operatorname{index}E=\sum_k([M_{p_k}:N_{p_k}]/\tau(p_k))$$

$$K_\tau^E(M|N)=\sum_k\{\tau(p_k)\log[M_{p_k}:N_{p_k}]+2\eta(\tau(p_k))\}.$$

We remark that the equality (ii) on index E is a well-known local index formula as described in [3], [6] and the equality (ii) on $K_\tau^E(M|N)$ is the same formula as that on $H_\tau(M|N)$. See Theorem 4.4 in [7].

The details of the present note will appear elsewhere.

References

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