84. Regular Elements of Abstract Affine Near-rings

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1. Introduction. In his paper [2], Steinfeld characterizes the regular elements of a ring in terms of quasi-ideals.

The purpose of this note is to extend the above result to a class of abstract affine near-rings. An example is given to show that the result does not hold for arbitrary near-rings.

For the basic terminology and notation we refer to [1].

2. Preliminaries. Let N be a near-ring, which always means right one throughout this note.

If A, B and C are three non-empty subsets of N, then AB (ABC) denotes the set of all finite sums of the form $\sum a_k b_k$ with $a_k \in A$, $b_k \in B$ ($\sum a_k b_k c_k$ with $a_k \in A$, $b_k \in B$, $c_k \in C$), and $A * B$ denotes the set of all finite sums of the form $\sum (a_k(a'_k + b_k) - a_k a'_k)$ with a_k, $a'_k \in A$, $b_k \in B$. Note that $ABC = (AB)C \subseteq A(BC)$ in general, and that $ABC = (AB)C = A(BC)$ if $A \subseteq N_d$, where N_d is the set of all distributive elements of N.

A *right N-subgroup* (*left N-subgroup*) of N is a subgroup H of $(N, +)$ such that $HN \subseteq H$ ($NH \subseteq H$). A *quasi-ideal* of N is a subgroup Q of $(N, +)$ such that $QN \cap NQ \cap N * Q \subseteq Q$. Right N-subgroups and left N-subgroups are quasi-ideals. The intersection of a family of quasi-ideals is again a quasi-ideal.

Lemma 1. *Let e be an idempotent element of a near-ring N, and let R be a right N-subgroup of N. Then the following assertions hold:*

(i) *$R(Ne) = Re$.*

(ii) *Re is a quasi-ideal of N such that $Re = R \cap Ne$.*

Proof. (i) We have $R(Ne) = RNe \subseteq Re$ and $Re = Ree \subseteq RNe = R(Ne)$. So $R(Ne) = Re$.

(ii) Since R and Ne are quasi-ideals of N, it suffices to prove the relation $Re = R \cap Ne$. As $Re \subseteq R \cap Ne$, we have to show only $R \cap Ne \subseteq Re$. Any element x of $R \cap Ne$ has the form $x = r = ne$ with $r \in R$, $n \in N$, whence $x = ne = nee = re \in Re$.

For an element n of a near-ring N, $(n)_r$ ($(n)_l$) denotes the right (left) N-subgroup of N generated by n, and $[n]$ denotes the subgroup of $(N, +)$ generated by n.

An element n of a near-ring N is called *regular* if $n = nxn$ for some element x of N.

Lemma 2. *Let n be a regular element of a near-ring N. Then the following assertions hold:*

(I) $(n)_r=[n]N$ *and* $(n)_l=Nn$.

(II) *There exist idempotent elements e, f in N such that* $(n)_r=(e)_r$ *and* $(n)_l=(f)_l$.

Proof. (I) By the regularity of the element n, $n\in[n]N$. Moreover $[n]N$ is a right N-subgroup of N. So $(n)_r\subseteq[n]N$. Since the inclusion $[n]N\subseteq(n)_r$ always holds, we have $(n)_r=[n]N$. Similarly $(n)_l=Nn$.

(II) As n is regular, $n=nxn$ for an x in N. Put $e=nx$, then $e^2=e$ and $n=en$. Hence $n\in[e]N$ and $e\in[n]N$. These results imply $[n]N=[e]N$. So, by (I), we have $(n)_r=(e)_r$. Similarly we have $(n)_l=(f)_l$ with $f^2=f=xn$.

3. Main result. A near-ring N is called an *abstract affine near-ring* if N is abelian and $N_0=N_d$, where N_0 is the zero-symmetric part of N.

Lemma 3. *Let a be an element of an abstract affine near-ring N. Then the following assertions hold:*

(A) $(a)_r=[a]+[a]N$ *and* $(a)_rN\subseteq[a]N$.

(B) $(a)_l=[a]+Na$ *and* $N(a)_l\subseteq Na$.

Proof. (A) is evident.

(B) The inclusion $[a]+Na\subseteq(a)_l$ always holds. On the other hand, we have

$$N([a]+Na)=(N_0+N_c)([a]+Na)=N_0[a]+N_0(Na)+N_c,$$

where N_c is the constant part of N. Moreover we have

$$N_0[a]+N_c=Na \quad\text{and}\quad N_0(Na)\subseteq Na.$$

Hence $N([a]+Na)\subseteq Na\subseteq[a]+Na$. Therefore $[a]+Na$ is a left N-subgroup of N containing the element a. So we have $(a)_l\subseteq[a]+Na$. Thus $(a)_l=[a]+Na$. The above argument shows also $N(a)_l\subseteq Na$.

For an element n of a near-ring N, we denote $n-n0$ by n_0 and $n0$ by n_c. Then $n=n_0+n_c$ with $n_0\in N_0$, $n_c\in N_c$.

It is easy to see that, for elements a, b of an abstract affine near-ring N, $(-a)_0=-a_0$, $(a+b)_0=a_0+b_0$ and $(ab)_0=a_0b_0$.

Lemma 4. *Let a be an element of an abstract affine near-ring N. Then a is regular if and only if $a\in[a]Na$.*

Proof. Suppose that $a\in[a]Na$. Then a has the form $a=\sum(i_ka)n_ka$ with integers i_k and $n_k\in N$. By the above remark, we have $a_0=\sum(i_ka_0)(n_k)_0a_0$. Put $m=\sum i_k(n_k)_0$, then $a_0=a_0ma_0$ and $m\in N_0$. Hence we have

$$a(m-ma_c)a=a(ma-ma_c)=a(ma_0+ma_c-ma_c)=ama_0$$

and

$$ama_0=(a_0+a_c)ma_0=a_0ma_0+a_c=a_0+a_c=a.$$

So $a(m-ma_c)a=a$. Thus a is regular.

Since the converse is evident, the proof is complete.

Now we are ready to state the main result of this note.

Theorem. *The following assertions concerning an element a of an abstract affine near-ring N are equivalent:*

(1) *a is regular.*

(2) $(a)_r(a)_l=(a)_r\cap(a)_l$.

(3) $(a)_r^2=(a)_r$, $(a)_l^2=(a)_l$ *and the product* $(a)_r(a)_l$ *is a quasi-ideal of* N.

Proof. (1)$\Rightarrow$(3): By Lemma 2, we have $(a)_r=(e)_r$ with a suitable idempotent element e of N. Then $e=e^2\in(e)_r^2$, and $(e)_r^2$ is a right N-subgroup of N. Hence

$$(a)_r=(e)_r\subseteq(e)_r^2=(a)_r^2\subseteq(a)_r,$$

that is, $(a)_r=(a)_r^2$. Similarly $(a)_l=(a)_l^2$.

Again by Lemma 2, we have $(a)_l=(f)_l=Nf$ with a suitable idempotent element f of N. So $(a)_r(a)_l=(a)_r(Nf)$. Hence, by Lemma 1, the product $(a)_r(a)_l$ is a quasi-ideal of N.

(3)$\Rightarrow$(2): First we show that $(a)_r\cap(a)_l\subseteq(a)_r(a)_l$. The condition (3) and Lemma 3 imply

$$(a)_r=(a)_r^2\subseteq(a)_rN\subseteq[a]N\subseteq(a)_r,$$

whence $(a)_r=[a]N$. Similarly $(a)_l=Na$. So, by [1, Proposition 9.73], we have

$$(a)_r=a_0N_0+[a]N\cap N_c \quad\text{and}\quad (a)_l=N_0a_0+N_c. \tag{3.1}$$

From these relations, it follows easily that

$$(a)_r\cap(a)_l=a_0N_0\cap N_0a_0+[a]N\cap N_c$$

and

$$(a)_r(a)_l=(a_0N_0)(N_0a_0)+[a]N\cap N_c.$$

Hence it suffices to show that $a_0N_0\cap N_0a_0\subseteq(a_0N_0)(N_0a_0)$.

Now, from the relations (3.1), it follows also that

$$(a)_r^2=(a_0N_0)^2+[a]N\cap N_c \quad\text{and}\quad (a)_l^2=(N_0a_0)^2+N_c. \tag{3.2}$$

The relations (3.1), (3.2) and the condition (3) imply that $a_0N_0=(a_0N_0)^2$ and $N_0a_0=(N_0a_0)^2$. Hence we have

$$a_0N_0=(a_0N_0)^3\subseteq(a_0N_0)N_0(a_0N_0)=(a_0N_0)(N_0a_0)N_0$$

and

$$N_0a_0=(N_0a_0)^3\subseteq(N_0a_0)N_0(N_0a_0)=N_0(a_0N_0)(N_0a_0),$$

whence

$$a_0N_0\cap N_0a_0\subseteq(a_0N_0)(N_0a_0)N_0\cap N_0(a_0N_0)(N_0a_0). \tag{3.3}$$

On the other hand, since $(a)_r(a)_l$ is a quasi-ideal of N, and since $(a)_r(a)_l\cap N_0=(a_0N_0)(N_0a_0)$, by [3, Proposition 2] the product $(a_0N_0)(N_0a_0)$ is a quasi-ideal of N_0. So, by [3, Proposition 3], we have

$$(a_0N_0)(N_0a_0)N_0\cap N_0(a_0N_0)(N_0a_0)\subseteq(a_0N_0)(N_0a_0). \tag{3.4}$$

The relations (3.3) and (3.4) imply $a_0N_0\cap N_0a_0\subseteq(a_0N_0)(N_0a_0)$. Thus we have $(a)_r\cap(a)_l\subseteq(a)_r(a)_l$.

Since the inclusion $(a)_r(a)_l\subseteq(a)_r\cap(a)_l$ always holds, the proof is complete.

(2)$\Rightarrow$(1): The condition (2) and Lemma 3 imply that

$$a\in(a)_r\cap(a)_l=(a)_r(a)_l\subseteq(a)_rN\subseteq[a]N,$$

whence $(a)_r=[a]N$. Similarly $(a)_l=Na$. So $a\in([a]N)(Na)\subseteq[a]Na$. Thus, by Lemma 4, a is regular.

4. Remarks. The implications (1)$\Rightarrow$(2) and (1)$\Rightarrow$(3) hold for arbitrary near-rings. However, the following example shows that neither the assertion (2) nor (3) implies the assertion (1) in general.

Let $N=\{0, a, b, c\}$ be the near-ring due to [1, Near-rings of low order (E-21)] defined by the tables

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	b	0
c	a	a	c	a .

Then $(c)_r=(c)_l=N$. So, the assertions (2) and (3) hold for the element c. But c is not regular, because $cxc=a$ for all elements x of N.

References

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[3] I. Yakabe: Quasi-ideals in near-rings. Math. Rep. Kyushu Univ., **14**, 41–46 (1983).